Ranges of functors in algebra

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General framework

- We are given categories $\mathcal{A}$, $\mathcal{B}$, $\mathcal{S}$ together with functors $\Phi\colon \mathcal{A} \to \mathcal{S}$ and $\Psi\colon \mathcal{B} \to \mathcal{S}$.

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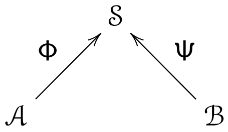

Hochster's Theorem for commutative unital rings

Stone duality for bounded distributive lattices

The Congruence Lattice Problem

From unliftable diagrams to non-representability

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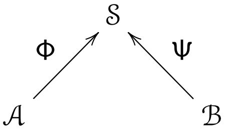

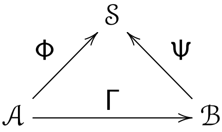

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- Hence we need an assumption of the form "for many $A \in \mathcal{A}$, there exists $B \in \mathcal{B}$ such that $\Phi(A) \cong \Psi(B)$".

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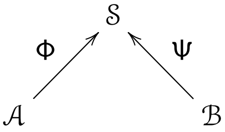

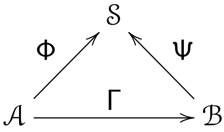

- Hence we need an assumption of the form "for many $A \in \mathcal{A}$, there exists $B \in \mathcal{B}$ such that $\Phi(A) \cong \Psi(B)$".
- Ask for $\Gamma\colon A \mapsto B$ to be a functor (at least on a large enough subcategory of $\mathcal{A}$).

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- A proper ideal P in a commutative, unital ring A is prime if A/P is a domain. Equivalently, $xy \in P \Rightarrow (x \in P$ or $y \in P)$, for all $x, y \in A$.

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- Is there an intrinsic characterization of the topological spaces of the form $\operatorname{Spec} A$?

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- Spec A is a spectral space, for every commutative unital ring A (well known and easy).

Hochster's Theorem

The converse of the above observation holds:

Theorem (Hochster 1969)

Every spectral space X is homeomorphic to $\operatorname{Spec} A$ for some commutative unital ring A.

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- In order for that observation to make sense, the morphisms need to be specified.
- On the ring side, just consider unital ring homomorphisms.
- On the spectral space side, consider surjective spectral maps. For spectral spaces X and Y, a map $f \colon X \to Y$ is spectral if $f^{-1}[V] \in \overset{\circ}{\mathcal{K}}(X)$ whenever $V \in \overset{\circ}{\mathcal{K}}(Y)$.

Bounded distributive lattices

- A lattice is a structure $(L, \vee, \wedge)$, where $\vee$ and $\wedge$ are both binary operations on a set L such that there is a partial ordering $\leq$ for which $x \vee y = \sup(x, y)$ (the join of $\{x, y\}$) and $x \wedge y = \inf(x, y)$ (the meet of $\{x, y\}$) $\forall x, y \in L$.

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- For a bounded distributive lattice D, $\operatorname{Spec} D$ is defined as for rings, on the prime ideals of D. It is a spectral space.

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- For bounded distributive lattices D and E and a $0,1$-lattice homomorphism $f\colon D \to E$, the map $\operatorname{Spec} f\colon \operatorname{Spec} E \to \operatorname{Spec} D,\ Q \mapsto f^{-1}[Q]$ is spectral.

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- For spectral spaces X and Y and a spectral map $\varphi\colon X \to Y$, the map $\overset{\circ}{\mathcal{K}}(\varphi)\colon \overset{\circ}{\mathcal{K}}(Y) \to \overset{\circ}{\mathcal{K}}(X)$, $V \mapsto \varphi^{-1}[V]$ is a $0,1$-lattice homomorphism.

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The pair $(\operatorname{Spec}, \overset{\circ}{\mathcal{K}})$ induces a (categorical) duality, between bounded distributive lattices with $0,1$-lattice homomorphisms and spectral spaces with spectral maps.

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Note that in Hochster's Theorem's case, we do not obtain a duality (a ring is not determined by its spectrum).

Congruences of a lattice

- A congruence of a lattice $(L, \vee, \wedge)$ is an equivalence relation θ on L such that $x_1 \equiv_\theta y_1$ and $x_2 \equiv_\theta y_2$ implies both $x_1 \vee x_2 \equiv_\theta y_1 \vee y_2$ and $x_1 \wedge x_2 \equiv_\theta y_1 \wedge y_2$ (we say that θ is compatible with both operations $\vee$ and $\wedge$).

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- The concept of congruence can be extended to any "universal algebra" (i.e., nonempty set A with a collection of operations $A^n \to A$ for various n).
- For example, the congruences of a group G are in one-to-one correspondence with the normal subgroups of G.
- However, the congruences of a lattice L are, usually, not in any natural one-to-one correspondence with subsets of L.

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- The set Con L of all congruences of a lattice L, partially ordered under $\subseteq$, is a complete lattice, in which

$$\bigwedge_{i\in I}\theta_i = \bigcap_{i\in I}\theta_i\,,$$

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- A congruence θ is finitely generated iff it is a compact element of Con L, that is, whenever $\theta \subseteq \bigvee_{i\in I}\theta_i$, there exists a finite subset J of I such that $\theta \subseteq \bigvee_{i\in J}\theta_i$.

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- The lattice Con L is algebraic, that is, it is complete and every congruence is $\bigvee_{i \in I} \theta_i$ with compact θ_i.

The congruence lattice of a lattice (cont'd)

The algebraicity of the lattices $\operatorname{Con} L$ is not lattice-specific: it holds for any universal algebra (e.g., group, module, ring. . .).

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- Funayama and Nakayama's Theorem is a very important property of lattices. It does not extend to groups, modules, rings... For example, $A \cap (B + C) \neq (A \cap B) + (A \cap C)$ for submodules A, B, C of a given module.

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- In the 1940's, R. P. Dilworth proved that conversely, every finite distributive lattice is the congruence lattice of a (finite) lattice.
- Then he asked whether this could be extended to the infinite case:

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CLP

The Congruence Lattice Problem (CLP); Dilworth, 1940's

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- It has to be noted that $\operatorname{Con}_{\mathrm{c}} L$ is not a lattice as a rule: for compact congruences α and β, the join $\alpha \vee \beta$ is compact, but the meet $\alpha \cap \beta$ may not be compact.
- Hence, $\operatorname{Con}_{\mathrm{c}} L$ is a $(\vee, 0)$-semilattice. It is distributive, that is, whenever $\alpha \subseteq \beta_1 \vee \beta_2$ in $\operatorname{Con}_{\mathrm{c}} L$, there are $\alpha_i \subseteq \beta_i$ in $\operatorname{Con}_{\mathrm{c}} L$ such that $\alpha = \alpha_1 \vee \alpha_2$.

Known positive instances of CLP

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Semilattice formulation of CLP

Is every distributive $(\vee, 0)$-semilattice representable, that is, isomorphic to $\mathrm{Con_c}\, L$ for some lattice L?

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Theorem

Let S be a distributive $(\vee, 0)$-semilattice. In each of the following cases, S is representable:

1. S is countable (Bauer $\sim$ 1980);
2. $\mathrm{card}\, S \leq \aleph_1$ (Huhn 1989);
3. S is a lattice (Schmidt 1981);
4. $S = \varinjlim_{n<\omega} S_n$, with all transition maps $S_n \to S_{n+1}$ $(\vee, 0)$-homomorphisms and all S_n lattices (W. 2003).

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Theorem (Růžička 2008; yields the optimal cardinality bound)

The distributive $(\vee, 0)$-semilattice $S_{\aleph_2}$ is not representable.

A heavy cube

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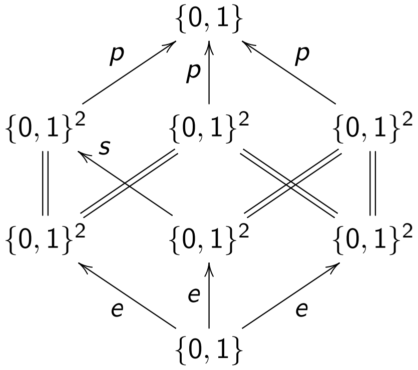
We consider the following commutative diagram $\mathcal{D}_c$ of $(\vee, 0)$-semilattices and $(\vee, 0)$-homomorphisms,



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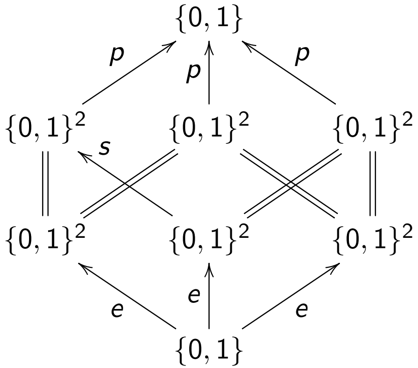

where $e(x) = (x, x)$, $p(x, y) = x \vee y$, and $s(x, y) = (y, x)$.

A heavy cube (cont'd)

Theorem (Tůma and W. 2001)

The cube $\mathcal{D}_c$ is not representable (with respect to the functor $\mathrm{Con_c}$), by any cube of sectionally complemented lattices and lattice homomorphisms.

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In fact, it turns out that the result above can be extended to a much broader algebraic context; in particular, it is not lattice-specific:

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For binary relations α and β on a set A, we set
$\alpha \circ \beta \underset{\text{def}}{=} \{(x, y) \in A \times A \mid (\exists z \in A)((x, z) \in \alpha \text{ and } (z, y) \in \beta)\}$.

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For example, groups, modules, rings are all congruence-permutable (e.g., $HK = KH$ for normal subgroups in a group).

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For example, groups, modules, rings are all congruence-permutable (e.g., $HK = KH$ for normal subgroups in a group). However, not all lattices are congruence-permutable (e.g., consider the three-element chain).

From heavy cubes to heavy semilattices

The proof of non-representability of $\mathcal{D}_c$, by congruence-permutable algebras, can be "converted" to the construction of a non-representable distributive $(\vee, 0)$-semilattice.

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In the case of sectionally complemented lattices, groups, modules, rings, the cardinality bound $\aleph_2$ is optimal. However, not every lattice is sectionally complemented. Hence, the negative solution to CLP was much trickier.

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- Hence, Murray - von Neumann equivalence classes $[a] \underset{\text{def}}{=} \{x \mid a \sim x\}$, for idempotent matrices a over R, can be added, *via* $[a] + [b] \underset{\text{def}}{=} [a \oplus b]$.

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- The monoid $\mathrm{V}(R) \underset{\text{def}}{=} \{[a] \mid a \text{ idempotent matrix on } R\}$ is commutative ($x + y = y + x$) and conical ($x + y = 0 \Rightarrow x = y = 0$).

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- If $x_1 \sim y_1$ and $x_2 \sim y_2$, then $x_1 \oplus x_2 \sim y_1 \oplus y_2$.
- Hence, Murray - von Neumann equivalence classes $[a] \underset{\text{def}}{=} \{x \mid a \sim x\}$, for idempotent matrices a over R, can be added, *via* $[a] + [b] \underset{\text{def}}{=} [a \oplus b]$.
- The monoid $V(R) \underset{\text{def}}{=} \{[a] \mid a \text{ idempotent matrix on } R\}$ is commutative ($x + y = y + x$) and conical ($x + y = 0 \Rightarrow x = y = 0$). It encodes the nonstable K_0-theory of R.

Representability with respect to nonstable K_0-theory

Theorem (Bergman 1974, Bergman and Dicks 1978)

Every commutative conical monoid is $V(R)$ for some ring R.

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Definition (Warfield 1972, Ara 1997)

A ring R is an exchange ring if for all $x \in R$, there are an idempotent $e \in R$ and $r, s \in R$ such that $e = rx = x + s - sx$.

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- This condition is left-right symmetric.
- Every von Neumann regular ring (i.e., satisfying $(\forall x)(\exists y)(xyx = x)$) is an exchange ring, and a C*-algebra is an exchange ring iff it has real rank zero (Ara, Goodearl, O'Meara, and Pardo 1998, Ara 1997).

Representability with respect to nonstable K_0-theory (cont'd)

Theorem (Ara 1997)

Let R be an exchange ring. Then $\mathrm{V}(R)$ is a refinement monoid, that is, for all $a_0, a_1, b_0, b_1 \in \mathrm{V}(R)$ such that $a_0 + a_1 = b_0 + b_1$, there are $c_{i,j} \in \mathrm{V}(R)$, for $i, j \in \{0, 1\}$, such that each $a_i = c_{i,0} + c_{i,1}$ and $b_i = c_{0,i} + c_{1,i}$.

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The converse is unknown:

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Problem

Does every conical refinement monoid appear as $V(R)$, for some exchange ring R?

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A monoid is simplicial if it is $\mathbb{N}^n$ for some nonnegative integer n.

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Theorem (W. 2013)

There is a commutative cube of simplicial monoids that can be lifted, with respect to the functor V, by exchange rings and by C*-algebras of real rank 1, but not by semiprimitive exchange rings, thus neither by von Neumann regular rings nor by C*-algebras of real rank 0.

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Under fairly general categorical conditions, non-representable diagrams can be turned (*via* the so-called condensate construction) to non-representable objects.

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CLL (Gillibert and Wehrung 2011)

Under fairly general categorical conditions, non-representable diagrams can be turned (*via* the so-called condensate construction) to non-representable objects.

The (quite complex) condensate construction turns a diagram to an object, but it may increase the cardinality.

A non-representable object in nonstable K_0-theory

An application of CLL to the cube above yields the following:

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Theorem (W. 2013)

There exists a unital exchange ring of cardinality $\aleph_3$ (resp., an $\aleph_3$-separable unital C*-algebra of real rank 1) R, such that $\mathrm{V}(R)$ is not isomorphic to $\mathrm{V}(B)$ for any ring B which is either a C*-algebra of real rank 0 or a von Neumann regular ring.

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To paraphrase this, the nonstable K_0-theory of exchange rings properly contains those of von Neumann regular rings and of C*-algebras of real rank zero.

Back to spectral spaces

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- The Zariski spectrum construction can be extended to various contexts, such as Abelian ℓ-groups (yielding the ℓ-spectrum) and partially ordered, commutative unital rings (yielding the real spectrum).

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- The Zariski spectrum construction can be extended to various contexts, such as Abelian ℓ-groups (yielding the ℓ-spectrum) and partially ordered, commutative unital rings (yielding the real spectrum).
- Tailoring the methods above (in particular, CLL) to that new context, further results can be obtained on ℓ-spectra and real spectra.

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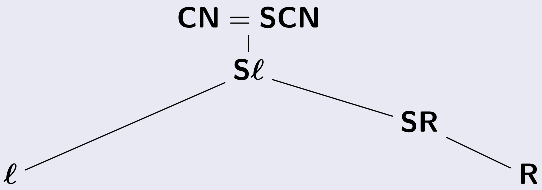
Theorem (W. 2017)

Let $\mathbf{CN} \underset{\text{def}}{=} \{$completely normal spectral spaces$\}$,
$\boldsymbol{\ell} \underset{\text{def}}{=} \{\ell$-spectra of Abelian ℓ-groups with unit$\}$,
$\mathbf{R} \underset{\text{def}}{=} \{$real spectra of commutative unital rings$\}$,
$\mathbf{SX} \underset{\text{def}}{=} \{$spectral subspaces of members of $\mathbf{X}\}$. Then



Thanks for your attention!