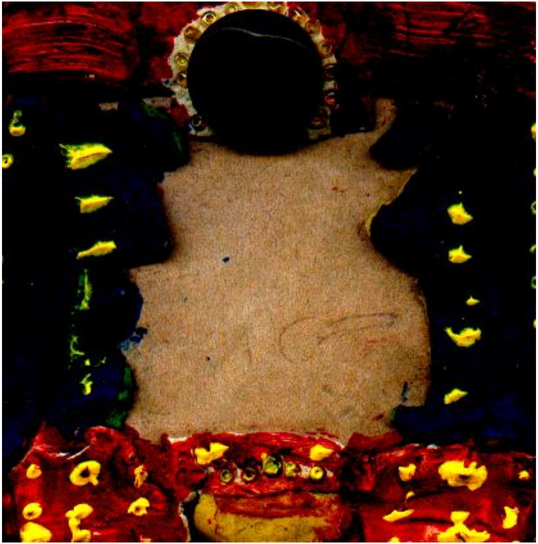
Spectra and subspectra arising from ℓ-groups and commutative rings

Friedrich Wehrung

Université de Caen
LMNO, CNRS UMR 6139
Département de Mathématiques
14032 Caen cedex
E-mail: friedrich.wehrung01@unicaen.fr
URL: http://wehrungf.users.lmno.cnrs.fr

TACL 2022, June 2022

A picture for the problem



Basic definitions (wrt. ℓ-spectrum)

- A subset I, in an Abelian ℓ-group G, is an ℓ-ideal if it is an order-convex subgroup closed under $\vee$ (equivalently, $\wedge$).

- A subset I, in an Abelian ℓ-group G, is an ℓ-ideal if it is an order-convex subgroup closed under $\vee$ (equivalently, $\wedge$).
- It is prime if $I \neq G$ and $x \wedge y \in I \Rightarrow \{x, y\} \cap I \neq \varnothing$.

Basic definitions (wrt. ℓ-spectrum)

- A subset I, in an Abelian ℓ-group G, is an ℓ-ideal if it is an order-convex subgroup closed under $\vee$ (equivalently, $\wedge$).
- It is prime if $I \neq G$ and $x \wedge y \in I \Rightarrow \{x, y\} \cap I \neq \varnothing$.
- $\mathrm{Spec}_\ell\, G \overset{\text{def}}{=} \{\text{prime } \ell\text{-ideals of } G\}$, topologized by the closed sets the $\{P \in \mathrm{Spec}_\ell\, G \mid X \subseteq P\}$ for $X \subseteq G$ (hull-kernel topology).

Basic definitions (wrt. ℓ-spectrum)

- A subset I, in an Abelian ℓ-group G, is an ℓ-ideal if it is an order-convex subgroup closed under $\vee$ (equivalently, $\wedge$).
- It is prime if $I \neq G$ and $x \wedge y \in I \Rightarrow \{x, y\} \cap I \neq \varnothing$.
- $\mathrm{Spec}_\ell\, G \stackrel{\text{def}}{=} \{\text{prime } \ell\text{-ideals of } G\}$, topologized by the closed sets the $\{P \in \mathrm{Spec}_\ell\, G \mid X \subseteq P\}$ for $X \subseteq G$ (hull-kernel topology).
- The topological space $\mathrm{Spec}_\ell\, G$ is called the ℓ-spectrum of G.

Basic definitions (wrt. real spectrum)

- A subset, in a commutative unital ring A, is a cone if it is both an additive and a multiplicative submonoid of A, containing $\{x^2 \mid x \in A\}$.

- A subset, in a commutative unital ring A, is a cone if it is both an additive and a multiplicative submonoid of A, containing $\{x^2 \mid x \in A\}$.
- A cone P is prime if $A = P \cup (-P)$ and the "support" $P \cap (-P)$ is a prime ideal of the ring A.

Basic definitions (wrt. real spectrum)

- A subset, in a commutative unital ring A, is a cone if it is both an additive and a multiplicative submonoid of A, containing $\{x^2 \mid x \in A\}$.
- A cone P is prime if $A = P \cup (-P)$ and the "support" $P \cap (-P)$ is a prime ideal of the ring A. It follows that $-1 \notin P$ (otherwise $1 \in P \cap (-P)$).

Basic definitions (wrt. real spectrum)

- A subset, in a commutative unital ring A, is a cone if it is both an additive and a multiplicative submonoid of A, containing $\{x^2 \mid x \in A\}$.
- A cone P is prime if $A = P \cup (-P)$ and the "support" $P \cap (-P)$ is a prime ideal of the ring A. It follows that $-1 \notin P$ (otherwise $1 \in P \cap (-P)$).
- $\operatorname{Spec}_{\mathrm{r}} A \overset{\text{def}}{=} \{\text{prime cones of } A\}$, endowed with the topology generated by all open subsets $\{P \in \operatorname{Spec}_{\mathrm{r}} A \mid a \notin P\}$ for $a \in A$, and we call $\operatorname{Spec}_{\mathrm{r}} A$ the real spectrum of A.

Basic definitions (wrt. real spectrum)

- A subset, in a commutative unital ring A, is a cone if it is both an additive and a multiplicative submonoid of A, containing $\{x^2 \mid x \in A\}$.
- A cone P is prime if $A = P \cup (-P)$ and the "support" $P \cap (-P)$ is a prime ideal of the ring A. It follows that $-1 \notin P$ (otherwise $1 \in P \cap (-P)$).
- $\operatorname{Spec}_{\mathrm{r}} A \overset{\text{def}}{=} \{\text{prime cones of } A\}$, endowed with the topology generated by all open subsets $\{P \in \operatorname{Spec}_{\mathrm{r}} A \mid a \notin P\}$ for $a \in A$, and we call $\operatorname{Spec}_{\mathrm{r}} A$ the real spectrum of A.
- $\operatorname{Spec}_{\mathrm{r}} A$ is homeomorphic to the Zariski spectrum of the real closure (Schwartz 1989) of the ring A.

Basic definitions (spectral spaces)

- Specialization preorder on a topological space X: $x \leqslant y$ if $y \in \overline{\{x\}}$.

Basic definitions (spectral spaces)

- Specialization preorder on a topological space X: $x \leqslant y$ if $y \in \overline{\{x\}}$.
- A topological space X is spectral if it is T_0 (i.e., $\leqslant$ is antisymmetric), every irreducible closed set is some $\overline{\{x\}}$, and $\overset{\circ}{\mathcal{K}}(X) \overset{\text{def}}{=} \{\text{compact open subsets of } X\}$ is a basis of open sets in X, closed under finite intersections (thus X is compact).

Basic definitions (spectral spaces)

- Specialization preorder on a topological space X: $x \leqslant y$ if $y \in \overline{\{x\}}$.
- A topological space X is spectral if it is T_0 (i.e., $\leqslant$ is antisymmetric), every irreducible closed set is some $\overline{\{x\}}$, and $\overset{\circ}{\mathcal{K}}(X) \overset{\text{def}}{=} \{\text{compact open subsets of } X\}$ is a basis of open sets in X, closed under finite intersections (thus X is compact).
- A spectral space X is completely normal if $(X, \leqslant)$ is a root system, that is, each $\overline{\{x\}}$ is a chain wrt $\leqslant$.

Basic definitions (spectral spaces)

- Specialization preorder on a topological space X: $x \leqslant y$ if $y \in \overline{\{x\}}$.
- A topological space X is spectral if it is T_0 (i.e., $\leqslant$ is antisymmetric), every irreducible closed set is some $\overline{\{x\}}$, and $\overset{\circ}{\mathcal{K}}(X) \overset{\text{def}}{=} \{\text{compact open subsets of } X\}$ is a basis of open sets in X, closed under finite intersections (thus X is compact).
- A spectral space X is completely normal if $(X, \leqslant)$ is a root system, that is, each $\overline{\{x\}}$ is a chain wrt $\leqslant$.

Proposition (Keimel 1971; Coste and Roy 1981)

All $\operatorname{Spec}_\ell G$, for an Abelian ℓ-group G with unit, and $\operatorname{Spec}_{\mathrm{r}} A$, for a commutative unital ring A, are completely normal spectral spaces.

Stone duality

- A map $f\colon X \to Y$ (between spectral spaces) is spectral if $f^{-1}[V] \in \mathring{\mathcal{K}}(X)$ whenever $V \in \mathring{\mathcal{K}}(Y)$.

Stone duality

- A map $f\colon X \to Y$ (between spectral spaces) is **spectral** if $f^{-1}[V] \in \overset{\circ}{\mathcal{K}}(X)$ whenever $V \in \overset{\circ}{\mathcal{K}}(Y)$. Hence spectral $\Rightarrow$ continuous.
- A **spectral subspace** of Y is $X \subseteq Y$ such that the inclusion map $X \hookrightarrow Y$ is spectral.

Stone duality

- A map $f\colon X \to Y$ (between spectral spaces) is spectral if $f^{-1}[V] \in \mathring{\mathcal{K}}(X)$ whenever $V \in \mathring{\mathcal{K}}(Y)$. Hence spectral $\Rightarrow$ continuous.
- A spectral subspace of Y is $X \subseteq Y$ such that the inclusion map $X \hookrightarrow Y$ is spectral.

Theorem (Stone 1933)

The category of all spectral spaces, with spectral maps, is dual to the category of all bounded distributive lattices, with $0, 1$-lattice homomorphisms.

Stone duality

- A map $f\colon X \to Y$ (between spectral spaces) is spectral if $f^{-1}[V] \in \mathring{\mathcal{K}}(X)$ whenever $V \in \mathring{\mathcal{K}}(Y)$. Hence spectral $\Rightarrow$ continuous.
- A spectral subspace of Y is $X \subseteq Y$ such that the inclusion map $X \hookrightarrow Y$ is spectral.

Theorem (Stone 1933)

The category of all spectral spaces, with spectral maps, is dual to the category of all bounded distributive lattices, with $0, 1$-lattice homomorphisms.

- Extended to generalized spectral spaces, with spectral maps, and distributive 0-lattices, with cofinal 0-lattice homomorphisms (Rump and Yang 2009).

Stone duality (cont'd)

- The dual of a spectral space X is the lattice $\mathring{\mathcal{K}}(X) \overset{\text{def}}{=} \{\text{compact opens of } X\}$.

Stone duality (cont'd)

- The dual of a spectral space X is the lattice $\overset{\circ}{\mathcal{K}}(X) \overset{\text{def}}{=} \{\text{compact opens of } X\}$.
- The dual of a bounded distributive lattice D is $\operatorname{Spec} D \overset{\text{def}}{=} \{\text{prime ideals of D}\}$.

Stone duality (cont'd)

- The dual of a spectral space X is the lattice $\overset{\circ}{\mathcal{K}}(X) \overset{\text{def}}{=} \{\text{compact opens of } X\}$.
- The dual of a bounded distributive lattice D is $\operatorname{Spec} D \overset{\text{def}}{=} \{\text{prime ideals of D}\}$.
- Spectral subspaces are dual to surjective lattice homomorphisms.

Statement of the problem

- For a class $\mathbf{X}$ of spectral spaces, denote by $\mathbf{SX}$ the class of all spectral subspaces of members of $\mathbf{X}$.

Statement of the problem

- For a class **X** of spectral spaces, denote by **SX** the class of all spectral subspaces of members of **X**.
- **CN** $\stackrel{\text{def}}{=}$ {completely normal spectral spaces}.

Statement of the problem

- For a class **X** of spectral spaces, denote by **SX** the class of all spectral subspaces of members of **X**.
- **CN** $\stackrel{\text{def}}{=}$ {completely normal spectral spaces}.
- $\ell \stackrel{\text{def}}{=} \{X \mid (\exists G \text{ Abelian } \ell\text{-group})(X \cong \operatorname{Spec}_\ell G)\}$.

Statement of the problem

- For a class **X** of spectral spaces, denote by **SX** the class of all spectral subspaces of members of **X**.
- **CN** $\overset{\text{def}}{=}$ {completely normal spectral spaces}.
- $\ell \overset{\text{def}}{=} \{X \mid (\exists G \text{ Abelian } \ell\text{-group})(X \cong \mathrm{Spec}_\ell\, G)\}$.
- **R** $\overset{\text{def}}{=} \{X \mid (\exists A \text{ commutative unital ring})(X \cong \mathrm{Spec}_\mathrm{r}\, A)\}$.

Statement of the problem

- For a class **X** of spectral spaces, denote by **SX** the class of all spectral subspaces of members of **X**.
- **CN** $\stackrel{\text{def}}{=}$ {completely normal spectral spaces}.
- $\ell \stackrel{\text{def}}{=} \{X \mid (\exists G \text{ Abelian } \ell\text{-group})(X \cong \operatorname{Spec}_\ell G)\}$.
- **R** $\stackrel{\text{def}}{=} \{X \mid (\exists A \text{ commutative unital ring})(X \cong \operatorname{Spec}_{\mathrm{r}} A)\}$.

Problem

Determine all possible containments and non-containments between **CN** = **SCN**, ℓ, **S**ℓ, **R**, **SR**, in every cardinality (i.e., according to card $\mathring{\mathcal{K}}(X)$).

$\subseteq$ between **CN**, ℓ, **R**, **S**ℓ, **SR**: the SPANNER

$\kappa \stackrel{\text{def}}{=} \operatorname{card} \overset{\circ}{\mathcal{K}}(X)$; red line $\leftrightharpoons$ sharp bound;

$\subseteq$ between **CN**, ℓ, **R**, **S**ℓ, **SR**: the SPANNER

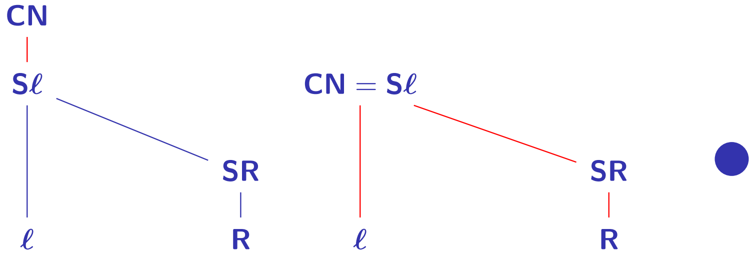
$\kappa \stackrel{\text{def}}{=} \operatorname{card} \mathring{\mathcal{K}}(X)$; red line $\leftrightharpoons$ sharp bound;

● = black hole (**SR** $=$ ℓ $=$ **R** $=$ **S**ℓ $=$ **SR**)

$\subseteq$ between **CN**, ℓ, **R**, **S**ℓ, **SR**: the SPANNER

$\kappa \stackrel{\text{def}}{=} \operatorname{card} \mathring{\mathcal{K}}(X)$; red line $\leftrightharpoons$ sharp bound;

● = black hole (**SR** = ℓ = **R** = **S**ℓ = **SR**)



$\kappa \geq \aleph_2$ $\kappa = \aleph_1$ $\kappa \leq \aleph_0$

Preliminary steps (Stone duality)

- Using Stone duality, we reduce everything to problems about bounded distributive lattices (and bounded homomorphisms).

Preliminary steps (Stone duality)

- Using Stone duality, we reduce everything to problems about bounded distributive lattices (and bounded homomorphisms).
- By Monteiro (1954), complete normality translates to the lattice-theoretical condition

$$(\forall a, b)(\exists x, y)(a \vee b = a \vee y = x \vee b \text{ and } x \wedge y = 0)\,.$$

Preliminary steps (Stone duality)

- Using Stone duality, we reduce everything to problems about bounded distributive lattices (and bounded homomorphisms).
- By Monteiro (1954), complete normality translates to the lattice-theoretical condition

$$(\forall a, b)(\exists x, y)(a \vee b = a \vee y = x \vee b \text{ and } x \wedge y = 0)\,.$$

- That property is obviously closed under homomorphic images. Hence, **CN** = **SCN**.

Preliminary steps (ℓ-spectrum)

- For an Abelian ℓ-group G, the Stone dual of $\mathrm{Spec}_\ell\, G$ is the distributive 0-lattice $\mathrm{Id}_{\mathrm{c}}^{\ell}\, G = \{\langle a\rangle^{\ell} \mid a \in G^{+}\}$.

Preliminary steps (ℓ-spectrum)

- For an Abelian ℓ-group G, the Stone dual of $\mathrm{Spec}_\ell\, G$ is the distributive 0-lattice $\mathrm{Id}^\ell_{\mathrm{c}}\, G = \{\langle a\rangle^\ell \mid a \in G^+\}$.
- Here $\langle a\rangle^\ell = \{x \in G \mid (\exists n \in \mathbb{N})(|x| \leq na)\}$.

Preliminary steps (ℓ-spectrum)

- For an Abelian ℓ-group G, the Stone dual of $\mathrm{Spec}_\ell\, G$ is the distributive 0-lattice $\mathrm{Id}_{\mathrm{c}}^{\ell}\, G = \{\langle a\rangle^\ell \mid a \in G^+\}$.
- Here $\langle a\rangle^\ell = \{x \in G \mid (\exists n \in \mathbb{N})(|x| \leq na)\}$.
- Thus questions about $\mathrm{Spec}_\ell\, G$ translate to questions about lattices $\mathrm{Id}_{\mathrm{c}}^{\ell}\, G$, for Abelian ℓ-groups G.

Preliminary steps (Brumfiel spectrum)

- *F-rings* are lattice-ordered rings satisfying $(x \wedge y = 0 \text{ and } z \geq 0) \Rightarrow (x \wedge yz = x \wedge zy = 0)$.

Preliminary steps (Brumfiel spectrum)

- *F-rings* are lattice-ordered rings satisfying $(x \wedge y = 0 \text{ and } z \geq 0) \Rightarrow (x \wedge yz = x \wedge zy = 0)$.
- Here, all *f*-rings will be commutative and unital.

Preliminary steps (Brumfiel spectrum)

- *F-rings* are lattice-ordered rings satisfying $(x \wedge y = 0 \text{ and } z \geq 0) \Rightarrow (x \wedge yz = x \wedge zy = 0)$.
- Here, all *f*-rings will be commutative and unital.
- Points of the Brumfiel spectrum $\mathrm{Spec}_{\mathrm{B}}\, A$ of an *f*-ring A are ℓ-ideals P (i.e., both additive ℓ-ideals and ring ideals) that are also prime as ring ideals (thus also as ℓ-ideals).

Preliminary steps (Brumfiel spectrum)

- *F-rings* are lattice-ordered rings satisfying $(x \wedge y = 0 \text{ and } z \geq 0) \Rightarrow (x \wedge yz = x \wedge zy = 0)$.
- Here, all *f*-rings will be commutative and unital.
- Points of the Brumfiel spectrum $\mathrm{Spec}_{\mathrm{B}}\, A$ of an *f*-ring A are ℓ-ideals P (i.e., both additive ℓ-ideals and ring ideals) that are also prime as ring ideals (thus also as ℓ-ideals).
- The Stone dual of $\mathrm{Spec}_{\mathrm{B}}\, A$ is the lattice $\mathrm{Id}^{\mathrm{r}}_{\mathrm{c}}\, A$ of all radical ℓ-ideals of A.

Preliminary steps (Brumfiel spectrum)

- *F-rings* are lattice-ordered rings satisfying $(x \wedge y = 0 \text{ and } z \geq 0) \Rightarrow (x \wedge yz = x \wedge zy = 0)$.
- Here, all *f*-rings will be commutative and unital.
- Points of the Brumfiel spectrum $\operatorname{Spec}_{\mathrm{B}} A$ of an *f*-ring A are ℓ-ideals P (i.e., both additive ℓ-ideals and ring ideals) that are also prime as ring ideals (thus also as ℓ-ideals).
- The Stone dual of $\operatorname{Spec}_{\mathrm{B}} A$ is the lattice $\mathrm{Id}_{\mathrm{c}}^{\mathrm{r}} A$ of all radical ℓ-ideals of A.
- Brumfiel spectra are the same as real spectra ($\operatorname{Spec}_{\mathrm{r}} A \cong \operatorname{Spec}_{\mathrm{B}} \mathrm{F}(A)$, where $\mathrm{F}(A)$ *f*-ring-envelope of A; $\operatorname{Spec}_{\mathrm{B}} A \cong \{Q \in \operatorname{Spec}_{\mathrm{r}} A \mid A^+ \subseteq Q\}$ *via* $P \mapsto A^+ + P$, closed subspace of a real spectrum, thus a real spectrum).

Preliminary steps (Brumfiel spectrum)

- *F-rings* are lattice-ordered rings satisfying $(x \wedge y = 0 \text{ and } z \geq 0) \Rightarrow (x \wedge yz = x \wedge zy = 0)$.
- Here, all *f*-rings will be commutative and unital.
- Points of the Brumfiel spectrum $\operatorname{Spec}_{\mathrm{B}} A$ of an *f*-ring A are ℓ-ideals P (i.e., both additive ℓ-ideals and ring ideals) that are also prime as ring ideals (thus also as ℓ-ideals).
- The Stone dual of $\operatorname{Spec}_{\mathrm{B}} A$ is the lattice $\mathrm{Id}_{\mathrm{c}}^{\mathrm{r}} A$ of all radical ℓ-ideals of A.
- Brumfiel spectra are the same as real spectra ($\operatorname{Spec}_{\mathrm{r}} A \cong \operatorname{Spec}_{\mathrm{B}} \mathrm{F}(A)$, where $\mathrm{F}(A)$ *f*-ring-envelope of A; $\operatorname{Spec}_{\mathrm{B}} A \cong \{Q \in \operatorname{Spec}_{\mathrm{r}} A \mid A^+ \subseteq Q\}$ *via* $P \mapsto A^+ + P$, closed subspace of a real spectrum, thus a real spectrum).
- Thus questions about real spectra translate to questions about lattices $\mathrm{Id}_{\mathrm{c}}^{\mathrm{r}} A$, for *f*-rings A.

The other trivial spanner containment

- It is $\mathbf{SR} \subseteq \mathbf{S}\ell$ (equivalently, $\mathbf{R} \subseteq \mathbf{S}\ell$).

The other trivial spanner containment

- It is $\mathbf{SR} \subseteq \mathbf{S}\ell$ (equivalently, $\mathbf{R} \subseteq \mathbf{S}\ell$).
- This means that every $\mathrm{Id}_c^r A$ (for a f-ring A) is a homomorphic image of $\mathrm{Id}_c^\ell G$ for some Abelian ℓ-group G with unit.

The other trivial spanner containment

- It is $\mathbf{SR} \subseteq \mathbf{S}\ell$ (equivalently, $\mathbf{R} \subseteq \mathbf{S}\ell$).
- This means that every $\mathrm{Id}^{\mathrm{r}}_{\mathrm{c}} A$ (for a *f*-ring A) is a homomorphic image of $\mathrm{Id}^{\ell}_{\mathrm{c}} G$ for some Abelian ℓ-group G with unit.
- Take $G \overset{\text{def}}{=} \{x \in A \mid (\exists n \in \mathbb{N})(|x| \leq n \cdot 1)\}$ with induced ℓ-group structure, and $\mathrm{Id}^{\ell}_{\mathrm{c}} G \twoheadrightarrow \mathrm{Id}^{\mathrm{r}}_{\mathrm{c}} A$, $\langle a \rangle^{\ell} \mapsto \langle a \rangle^{\mathrm{r}}$.

Condensates for one arrow: a basis for a few spanner non-containments

- We are given a homomorphism $\varphi\colon A \to B$ of first-order structures (over a vocabulary $\mathbf{v}$).

Condensates for one arrow: a basis for a few spanner non-containments

- We are given a homomorphism $\varphi \colon A \to B$ of first-order structures (over a vocabulary $\mathbf{v}$).
- The condensate construction on φ means to concentrate in a single object the "repetition" of φ, κ times where κ is an infinite cardinal.

Condensates for one arrow: a basis for a few spanner non-containments

- We are given a homomorphism $\varphi\colon A \to B$ of first-order structures (over a vocabulary $\mathbf{v}$).
- The condensate construction on φ means to concentrate in a single object the "repetition" of φ, κ times where κ is an infinite cardinal.
- Formally, $\mathrm{Cond}(\varphi, \kappa)$ is the $\mathbf{v}$-structure with universe

$$\{(x, y) \in A \times B^{\kappa} \mid y \text{ is almost constant and } y_{\infty} = \varphi(x)\}\,.$$

Condensates for one arrow: a basis for a few spanner non-containments

- We are given a homomorphism $\varphi\colon A \to B$ of first-order structures (over a vocabulary $\mathbf{v}$).
- The condensate construction on φ means to concentrate in a single object the "repetition" of φ, κ times where κ is an infinite cardinal.
- Formally, $\mathrm{Cond}(\varphi, \kappa)$ is the $\mathbf{v}$-structure with universe

$$\{(x, y) \in A \times B^{\kappa} \mid y \text{ is almost constant and } y_\infty = \varphi(x)\}\,.$$

- Under quite general conditions, if κ is large enough and the *arrow* φ is not representable wrt a given functor, then neither is the *object* $\mathrm{Cond}(\varphi, \kappa)$.

Applications to $\ell \subsetneq \mathbf{S}\ell$ and $\mathbf{R} \subsetneq \mathbf{SR}$

- The maps φ will be $0,1$-homomorphisms between bounded distributive lattices, best described by their Birkhoff dual maps (here, isotone maps between finite chains).

Applications to $\ell \subsetneqq \mathbf{S}\ell$ and $\mathbf{R} \subsetneqq \mathbf{SR}$

- The maps φ will be $0,1$-homomorphisms between bounded distributive lattices, best described by their Birkhoff dual maps (here, isotone maps between finite chains).
- For $\ell \subsetneqq \mathbf{S}\ell$, consider $\mathrm{Cond}(\varphi, \omega_1)$ where φ is the dual map of $\{1\} \to \{1,2\}$, $1 \mapsto 1$ (*not closed*).

Applications to $\boldsymbol{\ell} \subsetneq \mathbf{S}\boldsymbol{\ell}$ and $\mathbf{R} \subsetneq \mathbf{SR}$

- The maps φ will be $0,1$-homomorphisms between bounded distributive lattices, best described by their Birkhoff dual maps (here, isotone maps between finite chains).
- For $\boldsymbol{\ell} \subsetneq \mathbf{S}\boldsymbol{\ell}$, consider $\mathrm{Cond}(\varphi, \omega_1)$ where φ is the dual map of $\{1\} \to \{1,2\}$, $1 \mapsto 1$ (*not closed*).
- For $\mathbf{R} \subsetneq \mathbf{SR}$, consider $\mathrm{Cond}(\varphi, \omega_1)$ where φ is the dual map of $\{1,2\} \to \{1,2,3\}$, $1 \mapsto 1$, $2 \mapsto 3$ (*not convex*).

Applications to $\boldsymbol{\ell} \subsetneqq \mathbf{S}\boldsymbol{\ell}$ and $\mathbf{R} \subsetneqq \mathbf{SR}$

- The maps φ will be $0, 1$-homomorphisms between bounded distributive lattices, best described by their Birkhoff dual maps (here, isotone maps between finite chains).
- For $\boldsymbol{\ell} \subsetneqq \mathbf{S}\boldsymbol{\ell}$, consider $\mathrm{Cond}(\varphi, \omega_1)$ where φ is the dual map of $\{1\} \to \{1, 2\}$, $1 \mapsto 1$ (*not closed*).
- For $\mathbf{R} \subsetneqq \mathbf{SR}$, consider $\mathrm{Cond}(\varphi, \omega_1)$ where φ is the dual map of $\{1, 2\} \to \{1, 2, 3\}$, $1 \mapsto 1$, $2 \mapsto 3$ (*not convex*). (*Solves, in the negative, a 2012 problem by Mellor and Tressl, asking whether a spectral subspace of a real spectrum is a real spectrum*).

An example for $\ell \not\subseteq$ **SR**

- Involves the lexicographical power $\mathbb{Z}\langle\omega_1^{\mathrm{op}}\rangle$ (a totally ordered Abelian group).

An example for $\boldsymbol{\ell} \not\subseteq \mathbf{SR}$

- Involves the lexicographical power $\mathbb{Z}\langle\omega_1^{\mathrm{op}}\rangle$ (a totally ordered Abelian group).
- The elements of $\mathbb{Z}\langle\omega_1^{\mathrm{op}}\rangle$ are finite linear combinations $x = \sum_{i<n} x_i \mathbf{t}^{\alpha_i}$ where each $x_i \in \mathbb{Z}$, each $\alpha_i < \omega_1$, and the indeterminate $\mathbf{t}$ is "infinitely small".

An example for $\ell \not\subseteq \mathbf{SR}$

- Involves the lexicographical power $\mathbb{Z}\langle\omega_1^{\mathrm{op}}\rangle$ (a totally ordered Abelian group).
- The elements of $\mathbb{Z}\langle\omega_1^{\mathrm{op}}\rangle$ are finite linear combinations $x = \sum_{i<n} x_i \mathbf{t}^{\alpha_i}$ where each $x_i \in \mathbb{Z}$, each $\alpha_i < \omega_1$, and the indeterminate $\mathbf{t}$ is "infinitely small".
- Then consider the Abelian ℓ-group F on generators a, b and relations $a, b \geq 0$.

An example for $\boldsymbol{\ell} \not\subseteq \mathbf{SR}$

- Involves the lexicographical power $\mathbb{Z}\langle\omega_1^{\mathrm{op}}\rangle$ (a totally ordered Abelian group).
- The elements of $\mathbb{Z}\langle\omega_1^{\mathrm{op}}\rangle$ are finite linear combinations $x = \sum_{i<n} x_i \mathbf{t}^{\alpha_i}$ where each $x_i \in \mathbb{Z}$, each $\alpha_i < \omega_1$, and the indeterminate $\mathbf{t}$ is "infinitely small".
- Then consider the Abelian ℓ-group F on generators a, b and relations $a, b \geq 0$.
- The desired counterexample is the Abelian ℓ-group $G \overset{\text{def}}{=} \mathbb{Z}\langle\omega_1^{\mathrm{op}}\rangle \times_{\mathrm{lex}} F$ (lexicographical product).

An example for $\ell \not\subseteq$ **SR**

- Involves the lexicographical power $\mathbb{Z}\langle\omega_1^{\mathrm{op}}\rangle$ (a totally ordered Abelian group).
- The elements of $\mathbb{Z}\langle\omega_1^{\mathrm{op}}\rangle$ are finite linear combinations $x = \sum_{i<n} x_i \mathbf{t}^{\alpha_i}$ where each $x_i \in \mathbb{Z}$, each $\alpha_i < \omega_1$, and the indeterminate $\mathbf{t}$ is "infinitely small".
- Then consider the Abelian ℓ-group F on generators a, b and relations $a, b \geq 0$.
- The desired counterexample is the Abelian ℓ-group $G \overset{\mathrm{def}}{=} \mathbb{Z}\langle\omega_1^{\mathrm{op}}\rangle \times_{\mathrm{lex}} F$ (lexicographical product).

Theorem (W 2017)

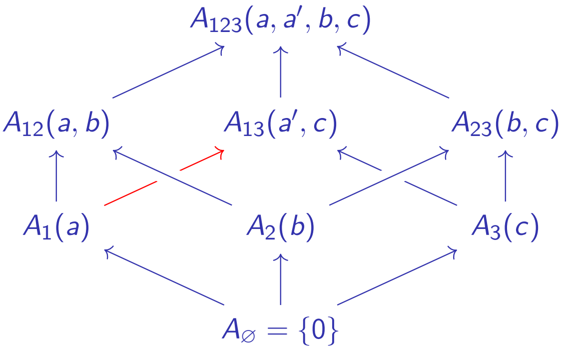
There is no commutative unital ring A such that $\mathrm{Spec}_\ell\, G$ is a spectral subspace of $\mathrm{Spec}_{\mathrm{r}}\, A$.

An example for $\mathbf{S}\ell \subsetneq \mathbf{CN}$ (and more, e.g. CBD)

Relies on a non-commutative diagram $\vec{A}$ of Abelian ℓ-groups:

An example for $\mathbf{S}\ell \subsetneq \mathbf{CN}$ (and more, e.g. CBD)

Relies on a non-commutative diagram $\vec{A}$ of Abelian ℓ-groups:



$0 \leq a \leq a' \leq 2a$; $b \geq 0$; $c \geq 0$.

All arrows inclusion maps, except $A_1(a) \to A_{13}(a', c)$ *via* $a \mapsto a'$.

- For every set I, $\mathrm{Id}_{\mathrm{c}}^{\ell} \vec{A}^{I}$ is a commutative diagram (indexed by $\{0,1\}^{3\times I}$) of completely normal distributive 0-lattices.

An example for $\mathbf{S}\boldsymbol{\ell} \subsetneq \mathbf{CN}$ (cont'd)

- For every set I, $\mathrm{Id}_{\mathrm{c}}^{\ell}\,\vec{A}^{I}$ is a commutative diagram (indexed by $\{0,1\}^{3\times I}$) of completely normal distributive 0-lattices.
- For every $\{0,1\}^3$-indexed commutative diagram $\vec{G}$ of Abelian ℓ-groups, $\mathrm{Id}_{\mathrm{c}}^{\ell}\,\vec{A} \ncong \mathrm{Id}_{\mathrm{c}}^{\ell}\,G$.

An example for $\mathbf{S}\ell \subsetneq \mathbf{CN}$ (cont'd)

- For every set I, $\mathrm{Id}_c^\ell \vec{A}^I$ is a commutative diagram (indexed by $\{0,1\}^{3\times I}$) of completely normal distributive 0-lattices.
- For every $\{0,1\}^3$-indexed commutative diagram $\vec{G}$ of Abelian ℓ-groups, $\mathrm{Id}_c^\ell \vec{A} \not\cong \mathrm{Id}_c^\ell G$.
- By using the condensate machinery (Gillibert and W 2011; *here not just for one arrow, but for the whole* $\{0,1\}^3$*-indexed diagram* $\mathrm{Id}_c^\ell \vec{A}$), this enables to construct

An example for $\mathbf{S}\ell \subsetneq \mathbf{CN}$ (cont'd)

- For every set I, $\mathrm{Id}_{\mathrm{c}}^{\ell}\,\vec{A}^I$ is a commutative diagram (indexed by $\{0,1\}^{3\times I}$) of completely normal distributive 0-lattices.
- For every $\{0,1\}^3$-indexed commutative diagram $\vec{G}$ of Abelian ℓ-groups, $\mathrm{Id}_{\mathrm{c}}^{\ell}\,\vec{A} \ncong \mathrm{Id}_{\mathrm{c}}^{\ell}\,G$.
- By using the condensate machinery (Gillibert and W 2011; *here not just for one arrow, but for the whole* $\{0,1\}^3$*-indexed diagram* $\mathrm{Id}_{\mathrm{c}}^{\ell}\,\vec{A}$), this enables to construct a completely normal distributive 0-lattice (very roughly speaking, "$\omega_2 \otimes \mathrm{Id}_{\mathrm{c}}^{\ell}\,\vec{A}$"), of cardinality $\aleph_2$, which is not a homomorphic image of $\mathrm{Id}_{\mathrm{c}}^{\ell}\,G$ for any Abelian ℓ-group G.

An example for $\mathbf{S}\ell \subsetneq \mathbf{CN}$ (cont'd)

- For every set I, $\mathrm{Id}^{\ell}_{\mathrm{c}} \vec{A}^{I}$ is a commutative diagram (indexed by $\{0,1\}^{3\times I}$) of completely normal distributive 0-lattices.
- For every $\{0,1\}^3$-indexed commutative diagram $\vec{G}$ of Abelian ℓ-groups, $\mathrm{Id}^{\ell}_{\mathrm{c}} \vec{A} \ncong \mathrm{Id}^{\ell}_{\mathrm{c}} G$.
- By using the condensate machinery (Gillibert and W 2011; *here not just for one arrow, but for the whole* $\{0,1\}^3$*-indexed diagram* $\mathrm{Id}^{\ell}_{\mathrm{c}} \vec{A}$), this enables to construct a completely normal distributive 0-lattice (very roughly speaking, "$\omega_2 \otimes \mathrm{Id}^{\ell}_{\mathrm{c}} \vec{A}$"), of cardinality $\aleph_2$, which is not a homomorphic image of $\mathrm{Id}^{\ell}_{\mathrm{c}} G$ for any Abelian ℓ-group G.
- Further extensions of the condensate construction (W 2021), together with Tuuri's Interpolation Theorem (1992), then make it possible to prove that $\mathrm{Id}^{\ell}_{\mathrm{c}} \mathcal{G} \stackrel{\mathrm{def}}{=} \{D \mid (\exists G \text{ Abelian } \ell\text{-group})(D \cong \mathrm{Id}^{\ell}_{\mathrm{c}} G)\}$ is not co-projective over $\mathscr{L}_{\infty\infty}$.

Moving towards

Moving towards ● (← I mean, the black one)

Theorem (W 2019)

Every (at most) countable completely normal distributive 0-lattice is isomorphic to $\mathrm{Id}_c^{\ell} G$ for some Abelian ℓ-group G with unit.

Moving towards ● ($\leftarrow$ I mean, the black one)

Theorem (W 2019)

Every (at most) countable completely normal distributive 0-lattice is isomorphic to $\mathrm{Id}_{\mathrm{c}}^{\ell}\, G$ for some Abelian ℓ-group G with unit.

- Hence every second countable, completely normal spectral space is homeomorphic to $\mathrm{Spec}_{\ell}\, G$ for some Abelian ℓ-group G with unit (i.e., "$\boldsymbol{\ell} = \mathbf{CN}$ on countable").

Moving towards ● (← I mean, the black one)

Theorem (W 2019)

Every (at most) countable completely normal distributive 0-lattice is isomorphic to $\operatorname{Id}_{\mathrm{c}}^{\ell} G$ for some Abelian ℓ-group G with unit.

- Hence every second countable, completely normal spectral space is homeomorphic to $\operatorname{Spec}_{\ell} G$ for some Abelian ℓ-group G with unit (i.e., "$\boldsymbol{\ell} = \mathbf{CN}$ on countable").
- In fact, G can be taken a vector lattice over any given countable totally ordered division ring $\mathbb{k}$ (ℓ-ideals then need be closed under scalar multiplication; the countability assumption on $\mathbb{k}$ cannot be dispensed with).

Idea of the proof ($\ell = \mathbf{CN}$ on countable)

- A lattice homomorphism $\varphi\colon A \to B$ is closed if whenever $a_0, a_1 \in A$ and $b \in B$, if $\varphi(a_0) \leq \varphi(a_1) \vee b$ then $\exists x \in A$ such that $a_0 \leq a_1 \vee x$ and $\varphi(x) \leq b$.

Idea of the proof ($\ell = \mathbf{CN}$ on countable)

- A lattice homomorphism $\varphi\colon A \to B$ is closed if whenever $a_0, a_1 \in A$ and $b \in B$, if $\varphi(a_0) \leq \varphi(a_1) \vee b$ then $\exists x \in A$ such that $a_0 \leq a_1 \vee x$ and $\varphi(x) \leq b$.
- For any ℓ-homomorphism $f\colon G \to H$ between ℓ-groups, the map $\mathrm{Id}_{\mathrm{c}}^{\ell} f\colon \mathrm{Id}_{\mathrm{c}}^{\ell} G \to \mathrm{Id}_{\mathrm{c}}^{\ell} H$, $\langle a \rangle^{\ell} \mapsto \langle f(a) \rangle^{\ell}$ is closed.

Idea of the proof ($\ell = \mathbf{CN}$ on countable)

- A lattice homomorphism $\varphi\colon A \to B$ is closed if whenever $a_0, a_1 \in A$ and $b \in B$, if $\varphi(a_0) \leq \varphi(a_1) \vee b$ then $\exists x \in A$ such that $a_0 \leq a_1 \vee x$ and $\varphi(x) \leq b$.
- For any ℓ-homomorphism $f\colon G \to H$ between ℓ-groups, the map $\mathrm{Id}_{\mathrm{c}}^{\ell} f\colon \mathrm{Id}_{\mathrm{c}}^{\ell} G \to \mathrm{Id}_{\mathrm{c}}^{\ell} H$, $\langle a \rangle^{\ell} \mapsto \langle f(a) \rangle^{\ell}$ is closed.
- Conversely, any surjective closed lattice homomorphism $\varphi\colon \mathrm{Id}_{\mathrm{c}}^{\ell} G \twoheadrightarrow D$ induces $\mathrm{Id}_{\mathrm{c}}^{\ell}(G/I) \cong D$ where $I \overset{\mathrm{def}}{=} \{x \in G \mid \varphi(\langle x \rangle^{\ell}) = 0\}$.

Idea of the proof ($\ell = \mathbf{CN}$ on countable)

- A lattice homomorphism $\varphi\colon A \to B$ is closed if whenever $a_0, a_1 \in A$ and $b \in B$, if $\varphi(a_0) \leq \varphi(a_1) \vee b$ then $\exists x \in A$ such that $a_0 \leq a_1 \vee x$ and $\varphi(x) \leq b$.
- For any ℓ-homomorphism $f\colon G \to H$ between ℓ-groups, the map $\mathrm{Id}_{\mathrm{c}}^{\ell} f\colon \mathrm{Id}_{\mathrm{c}}^{\ell} G \to \mathrm{Id}_{\mathrm{c}}^{\ell} H$, $\langle a \rangle^{\ell} \mapsto \langle f(a) \rangle^{\ell}$ is closed.
- Conversely, any surjective closed lattice homomorphism $\varphi\colon \mathrm{Id}_{\mathrm{c}}^{\ell} G \twoheadrightarrow D$ induces $\mathrm{Id}_{\mathrm{c}}^{\ell}(G/I) \cong D$ where $I \overset{\mathrm{def}}{=} \{x \in G \mid \varphi(\langle x \rangle^{\ell}) = 0\}$.
- Let $L = \{a_0, a_1, a_2 \ldots\}$ be a countable, completely normal bounded distributive lattice.

Idea of the proof ($\ell = \mathbf{CN}$ on countable)

- A lattice homomorphism $\varphi\colon A \to B$ is closed if whenever $a_0, a_1 \in A$ and $b \in B$, if $\varphi(a_0) \leq \varphi(a_1) \vee b$ then $\exists x \in A$ such that $a_0 \leq a_1 \vee x$ and $\varphi(x) \leq b$.
- For any ℓ-homomorphism $f\colon G \to H$ between ℓ-groups, the map $\mathrm{Id}^{\ell}_{\mathrm{c}} f\colon \mathrm{Id}^{\ell}_{\mathrm{c}} G \to \mathrm{Id}^{\ell}_{\mathrm{c}} H$, $\langle a\rangle^{\ell} \mapsto \langle f(a)\rangle^{\ell}$ is closed.
- Conversely, any surjective closed lattice homomorphism $\varphi\colon \mathrm{Id}^{\ell}_{\mathrm{c}} G \twoheadrightarrow D$ induces $\mathrm{Id}^{\ell}_{\mathrm{c}}(G/I) \cong D$ where $I \overset{\text{def}}{=} \{x \in G \mid \varphi(\langle x\rangle^{\ell}) = 0\}$.
- Let $L = \{a_0, a_1, a_2 \dots\}$ be a countable, completely normal bounded distributive lattice.
- Let $\mathrm{F}_{\ell}(\omega) \overset{\text{def}}{=}$ free Abelian ℓ-group on ω. It suffices to construct a surjective closed lattice homomorphism $\varphi\colon \mathrm{Id}^{\ell}_{\mathrm{c}} \mathrm{F}_{\ell}(\omega) \twoheadrightarrow L$ (because then, $L \cong \mathrm{Id}^{\ell}_{\mathrm{c}}(\mathrm{F}_{\ell}(\omega)/I)$ for a suitable ℓ-ideal I).

Idea of the proof ($\ell = \mathbf{CN}$ on countable, cont'd)

- Construct $\varphi\colon \operatorname{Id}_{\mathrm{c}}^{\ell} \mathrm{F}_{\ell}(\omega) \twoheadrightarrow L$, by iteratively defining an ascending sequence of $0,1$-lattice homomorphisms $\varphi_n\colon \operatorname{Op} \mathcal{F}_n \to L$ for suitable finite sublattices $\operatorname{Op} \mathcal{F}_n$ of $\operatorname{Id}_{\mathrm{c}}^{\ell} \mathrm{F}_{\ell}(\omega)$.

Idea of the proof ($\ell = \mathbf{CN}$ on countable, cont'd)

- Construct $\varphi\colon \operatorname{Id}_{\mathrm{c}}^{\ell} \mathrm{F}_{\ell}(\omega) \twoheadrightarrow L$, by iteratively defining an ascending sequence of $0,1$-lattice homomorphisms $\varphi_n\colon \operatorname{Op}\mathcal{F}_n \to L$ for suitable finite sublattices $\operatorname{Op}\mathcal{F}_n$ of $\operatorname{Id}_{\mathrm{c}}^{\ell} \mathrm{F}_{\ell}(\omega)$.
- For any $\mathcal{F} \subseteq \mathbb{Z}^{(\omega)}$, $\operatorname{Op}\mathcal{F}$ denotes the $0,1$-sublattice of $\mathfrak{P}(\mathbb{Z}^{(\omega)})$ generated by all $[\![f > 0]\!] \overset{\mathrm{def}}{=} \{x \in \mathbb{Z}^{(\omega)} \mid \langle f \mid x \rangle > 0\}$ where $f \in \mathcal{F} \cup (-\mathcal{F})$. Then set $\operatorname{Op}^{-}\mathcal{F} \overset{\mathrm{def}}{=} \operatorname{Op}\mathcal{F} \setminus \{\mathbb{Z}^{(\omega)}\}$.

Idea of the proof ($\ell = \mathbf{CN}$ on countable, cont'd)

- Construct $\varphi\colon \operatorname{Id}_{\mathrm{c}}^{\ell} \mathrm{F}_{\ell}(\omega) \twoheadrightarrow L$, by iteratively defining an ascending sequence of $0,1$-lattice homomorphisms $\varphi_n\colon \operatorname{Op} \mathcal{F}_n \to L$ for suitable finite sublattices $\operatorname{Op} \mathcal{F}_n$ of $\operatorname{Id}_{\mathrm{c}}^{\ell} \mathrm{F}_{\ell}(\omega)$.
- For any $\mathcal{F} \subseteq \mathbb{Z}^{(\omega)}$, $\operatorname{Op} \mathcal{F}$ denotes the $0,1$-sublattice of $\mathfrak{P}(\mathbb{Z}^{(\omega)})$ generated by all $[\![f > 0]\!] \overset{\text{def}}{=} \{x \in \mathbb{Z}^{(\omega)} \mid \langle f \mid x \rangle > 0\}$ where $f \in \mathcal{F} \cup (-\mathcal{F})$. Then set $\operatorname{Op}^{-} \mathcal{F} \overset{\text{def}}{=} \operatorname{Op} \mathcal{F} \setminus \{\mathbb{Z}^{(\omega)}\}$.
- By Baker-Beynon duality, $\operatorname{Id}_{\mathrm{c}}^{\ell} \mathrm{F}_{\ell}(\omega) \cong \operatorname{Op}^{-} \mathbb{Z}^{(\omega)}$.

Idea of the proof ($\ell = \mathbf{CN}$ on countable, cont'd)

- Construct $\varphi\colon \operatorname{Id}_{\mathrm{c}}^{\ell} \mathrm{F}_{\ell}(\omega) \twoheadrightarrow L$, by iteratively defining an ascending sequence of $0,1$-lattice homomorphisms $\varphi_n\colon \operatorname{Op}\mathcal{F}_n \to L$ for suitable finite sublattices $\operatorname{Op}\mathcal{F}_n$ of $\operatorname{Id}_{\mathrm{c}}^{\ell} \mathrm{F}_{\ell}(\omega)$.
- For any $\mathcal{F} \subseteq \mathbb{Z}^{(\omega)}$, $\operatorname{Op}\mathcal{F}$ denotes the $0,1$-sublattice of $\mathfrak{P}(\mathbb{Z}^{(\omega)})$ generated by all $[\![f > 0]\!] \overset{\text{def}}{=} \{x \in \mathbb{Z}^{(\omega)} \mid \langle f \mid x \rangle > 0\}$ where $f \in \mathcal{F} \cup (-\mathcal{F})$. Then set $\operatorname{Op}^{-}\mathcal{F} \overset{\text{def}}{=} \operatorname{Op}\mathcal{F} \setminus \{\mathbb{Z}^{(\omega)}\}$.
- By Baker-Beynon duality, $\operatorname{Id}_{\mathrm{c}}^{\ell} \mathrm{F}_{\ell}(\omega) \cong \operatorname{Op}^{-} \mathbb{Z}^{(\omega)}$.
- Let $\mathbb{Z}^{(\omega)} = \{f_n \mid n < \omega\}$.

Idea of the proof ($\ell = \mathbf{CN}$ on countable, cont'd)

- Construct $\varphi\colon \operatorname{Id}_{\mathrm{c}}^{\ell} \mathrm{F}_{\ell}(\omega) \twoheadrightarrow L$, by iteratively defining an ascending sequence of $0,1$-lattice homomorphisms $\varphi_n\colon \operatorname{Op}\mathcal{F}_n \to L$ for suitable finite sublattices $\operatorname{Op}\mathcal{F}_n$ of $\operatorname{Id}_{\mathrm{c}}^{\ell} \mathrm{F}_{\ell}(\omega)$.
- For any $\mathcal{F} \subseteq \mathbb{Z}^{(\omega)}$, $\operatorname{Op}\mathcal{F}$ denotes the $0,1$-sublattice of $\mathfrak{P}(\mathbb{Z}^{(\omega)})$ generated by all $[\![f > 0]\!] \overset{\text{def}}{=} \{x \in \mathbb{Z}^{(\omega)} \mid \langle f \mid x \rangle > 0\}$ where $f \in \mathcal{F} \cup (-\mathcal{F})$. Then set $\operatorname{Op}^{-}\mathcal{F} \overset{\text{def}}{=} \operatorname{Op}\mathcal{F} \setminus \{\mathbb{Z}^{(\omega)}\}$.
- By Baker-Beynon duality, $\operatorname{Id}_{\mathrm{c}}^{\ell} \mathrm{F}_{\ell}(\omega) \cong \operatorname{Op}^{-}\mathbb{Z}^{(\omega)}$.
- Let $\mathbb{Z}^{(\omega)} = \{f_n \mid n < \omega\}$.
- Given $\varphi_n\colon \operatorname{Op}\mathcal{F}_n \to L$, we find an extension $\varphi_{n+1}\colon \operatorname{Op}\mathcal{F}_{n+1} \to L$, with $\mathcal{F}_n \subseteq \mathcal{F}_{n+1}$, as follows.

Idea of the proof ($\ell = \mathbf{CN}$ on countable, cont'd 2)

- **Domain step**: if $n \equiv 0 \pmod 3$, then $\mathcal{F}_{n+1} = \mathcal{F}_n \cup \{f_{\lfloor n/3 \rfloor}\}$ and pick any extension $\varphi_{n+1} \colon \operatorname{Op} \mathcal{F}_{n+1} \to L$ (requires a nontrivial lattice-theoretical technical lemma for existence).

Idea of the proof ($\ell = \mathbf{CN}$ on countable, cont'd 2)

- **Domain step**: if $n \equiv 0 \pmod 3$, then $\mathcal{F}_{n+1} = \mathcal{F}_n \cup \{f_{\lfloor n/3 \rfloor}\}$ and pick any extension $\varphi_{n+1} \colon \operatorname{Op} \mathcal{F}_{n+1} \to L$ (requires a nontrivial lattice-theoretical technical lemma for existence).
- **Range step**: if $n \equiv 1 \pmod 3$, then $\mathcal{F}_{n+1} = \mathcal{F}_n \cup \{\delta_k\}$ for large enough k, then pick the extension $[\![\delta_k > 0]\!] \mapsto a_{\lfloor n/3 \rfloor}$, $[\![\delta_k < 0]\!] \mapsto 0$ (easy, because then $\operatorname{Op} \mathcal{F}_{n+1} \cong \operatorname{Op} \mathcal{F}_n * \mathsf{J}_2$).

Idea of the proof ($\ell = \mathbf{CN}$ on countable, cont'd 2)

- **Domain step**: if $n \equiv 0 \pmod 3$, then $\mathcal{F}_{n+1} = \mathcal{F}_n \cup \{f_{\lfloor n/3 \rfloor}\}$ and pick any extension $\varphi_{n+1} \colon \operatorname{Op} \mathcal{F}_{n+1} \to L$ (requires a nontrivial lattice-theoretical technical lemma for existence).
- **Range step**: if $n \equiv 1 \pmod 3$, then $\mathcal{F}_{n+1} = \mathcal{F}_n \cup \{\delta_k\}$ for large enough k, then pick the extension $[\![\delta_k > 0]\!] \mapsto a_{\lfloor n/3 \rfloor}$, $[\![\delta_k < 0]\!] \mapsto 0$ (easy, because then $\operatorname{Op} \mathcal{F}_{n+1} \cong \operatorname{Op} \mathcal{F}_n * \mathsf{J}_2$).
- **Closure step**: if $n \equiv 2 \pmod 3$, then $\mathcal{F}_{n+1}$ is a large enough finite subset of $\mathbb{Z}^{(\omega)}$ containing $\mathcal{F}_n$ such that all "closure defects" $\varphi_n(X) \leq \varphi_n(Y) \vee a_k$, where $X, Y \in \operatorname{Op} \mathcal{F}_n$ and $k \leq n$, are corrected in $\mathcal{F}_{n+1}$ (the hardest part of the argument).

Further feeding the 

Further feeding the (black) ●: $\mathbf{R} = \mathbf{CN}$ on countable

- Proceeds in a similar fashion as the argument for $\ell = \mathbf{CN}$ on countable, with more ingredients added. We fix a countable real-closed field $\Bbbk$.

Further feeding the (black) ●: $\mathbf{R} = \mathbf{CN}$ on countable

- Proceeds in a similar fashion as the argument for $\ell = \mathbf{CN}$ on countable, with more ingredients added. We fix a countable real-closed field $\mathbb{k}$.
- The basic features of the lattices $\operatorname{Op} \mathcal{F}$ need to be extended to the case where $\mathcal{F}$ consists of affine functionals, restricted to convex subsets of any $\mathbb{k}^d$.

Further feeding the (black) : $\mathbf{R} = \mathbf{CN}$ on countable

- Proceeds in a similar fashion as the argument for $\ell = \mathbf{CN}$ on countable, with more ingredients added. We fix a countable real-closed field $\Bbbk$.
- The basic features of the lattices $\operatorname{Op}\mathcal{F}$ need to be extended to the case where $\mathcal{F}$ consists of affine functionals, restricted to convex subsets of any $\Bbbk^d$.

Triangulation Theorem (Bochnak, Coste, and Roy 1987?)

Given semi-algebraic sets $S_0, \dots, S_l \subseteq S \subseteq \Bbbk^d$ with S closed bounded, there are a simplicial complex $\mathbb{K}$ in $\Bbbk^d$ and a semi-algebraic homeomorphism $\tau \colon S \to |\mathbb{K}|$ such that each $\tau[S_i]$ is partitioned (i.e., union of open simplices) by $\mathbb{K}$.

R = CN on countable (cont'd)

The following improvement of the Triangulation Theorem is needed:

$\mathbf{R} = \mathbf{CN}$ on countable (cont'd)

The following improvement of the Triangulation Theorem is needed:

Normal Triangulation Theorem (Baro 2010)

Let $\mathbb{K}$ be a simplicial complex of $\Bbbk^d$ and let $S_1, \dots, S_l$ be semi-algebraic subsets of $|\mathbb{K}|$. Then there are a triangulation $(\mathbb{L}, \psi)$ of $(|\mathbb{K}|; S_1, \dots, S_l)$ such that $\mathbb{L}$ is a subdivision of $\mathbb{K}$ and $\psi[S] = S$ for each open simplex of $\mathbb{K}$.

R = **CN** on countable (cont'd)

The following improvement of the Triangulation Theorem is needed:

Normal Triangulation Theorem (Baro 2010)

Let $\mathbb{K}$ be a simplicial complex of $\Bbbk^d$ and let $S_1, \ldots, S_l$ be semi-algebraic subsets of $|\mathbb{K}|$. Then there are a triangulation $(\mathbb{L}, \psi)$ of $(|\mathbb{K}|; S_1, \ldots, S_l)$ such that $\mathbb{L}$ is a subdivision of $\mathbb{K}$ and $\psi[S] = S$ for each open simplex of $\mathbb{K}$.

"Straightening up the semi-algebraic sets S_i while keeping the open simplices of $\mathbb{K}$ intact."

$\mathbf{R} = \mathbf{CN}$ on countable (cont'd)

The following improvement of the Triangulation Theorem is needed:

Normal Triangulation Theorem (Baro 2010)

Let $\mathbb{K}$ be a simplicial complex of $\Bbbk^d$ and let $S_1, \ldots, S_l$ be semi-algebraic subsets of $|\mathbb{K}|$. Then there are a triangulation $(\mathbb{L}, \psi)$ of $(|\mathbb{K}|; S_1, \ldots, S_l)$ such that $\mathbb{L}$ is a subdivision of $\mathbb{K}$ and $\psi[S] = S$ for each open simplex of $\mathbb{K}$.

"Straightening up the semi-algebraic sets S_i while keeping the open simplices of $\mathbb{K}$ intact."
Then the role of the lattices $\operatorname{Op}\mathcal{F}$ is played by images of lattices $\operatorname{Op}(\mathcal{F}, \Omega)$ (relativization of $\operatorname{Op}\mathcal{F}$ to a convex subset Ω) under semi-algebraic homeomorphisms. Induction step taken care of *via* the Normal Triangulation Theorem.

Stating $\mathbf{R} = \mathbf{CN}$ for countable (and more)

Theorem (W 2021)

Let $\mathbb{k}$ be a countable formally real field ($-1 \neq \sum_i x_i^2$). Then every countable completely normal bounded distributive lattice is isomorphic to $\mathrm{Id}_c^r A$ for some (commutative unital) f-ring and $\mathbb{k}$-algebra A.

Stating $\mathbf{R} = \mathbf{CN}$ for countable (and more)

Theorem (W 2021)

Let $\Bbbk$ be a countable formally real field ($-1 \neq \sum_i x_i^2$). Then every countable completely normal bounded distributive lattice is isomorphic to $\mathrm{Id}^{\mathrm{r}}_{\mathrm{c}}\, A$ for some (commutative unital) f-ring and $\Bbbk$-algebra A.

The countability of $\Bbbk$ cannot be dispensed with (W 2021).

Stating **R** = **CN** for countable (and more)

Theorem (W 2021)

Let $\Bbbk$ be a countable formally real field ($-1 \neq \sum_i x_i^2$). Then every countable completely normal bounded distributive lattice is isomorphic to $\mathrm{Id}^{\mathrm{r}}_{\mathrm{c}}\, A$ for some (commutative unital) *f*-ring and $\Bbbk$-algebra A.

The countability of $\Bbbk$ cannot be dispensed with (W 2021).

Corollary

Every second countable completely normal spectral space is homeomorphic to the real spectrum of some commutative unital ring.

The remaining identification: $\mathbf{CN} = \mathbf{S}\ell$ at $\aleph_1$

Theorem (Ploščica and W 2022)

Every completely normal bounded distributive lattice with $\leq \aleph_1$ elements is a homomorphic image of $\mathrm{Id}_{\mathrm{c}}^{\ell} G$ for some Abelian ℓ-group G with unit.

The remaining identification: $\mathbf{CN} = \mathbf{S}\ell$ at $\aleph_1$

Theorem (Ploščica and W 2022)

Every completely normal bounded distributive lattice with $\leq \aleph_1$ elements is a homomorphic image of $\mathrm{Id}_c^{\ell} G$ for some Abelian ℓ-group G with unit.

Again, this extends to vector lattices over countable totally ordered division rings $\Bbbk$. The countability of $\Bbbk$ cannot be dispensed with (W 2021).

The remaining identification: $\mathbf{CN} = \mathbf{S}\ell$ at $\aleph_1$

Theorem (Ploščica and W 2022)

Every completely normal bounded distributive lattice with $\leq \aleph_1$ elements is a homomorphic image of $\mathrm{Id}_c^{\ell} G$ for some Abelian ℓ-group G with unit.

Again, this extends to vector lattices over countable totally ordered division rings $\Bbbk$. The countability of $\Bbbk$ cannot be dispensed with (W 2021).

Corollary

Every completely normal spectral space, with $\leq \aleph_1$ compact open sets, can be embedded as a spectral subspace into $\mathrm{Spec}_{\ell} G$ for some Abelian ℓ-group G with unit.

Idea of the proof

- Write a completely normal bounded distributive lattice L with $\aleph_1$ elements as an ascending union $L = \bigcup(L_\xi \mid \xi < \omega_1)$ for countable completely normal bounded sublattices L_ξ.

Idea of the proof

- Write a completely normal bounded distributive lattice L with $\aleph_1$ elements as an ascending union $L = \bigcup(L_\xi \mid \xi < \omega_1)$ for countable completely normal bounded sublattices L_ξ.
- Iteratively represent all subdiagrams $(L_\xi \mid \xi < \alpha)$, for $\alpha < \omega_1$, as homomorphic images of $\Bbbk$-vector lattices diagrams ($\Bbbk$ given countable totally ordered division ring).

Idea of the proof

- Write a completely normal bounded distributive lattice L with $\aleph_1$ elements as an ascending union $L = \bigcup(L_\xi \mid \xi < \omega_1)$ for countable completely normal bounded sublattices L_ξ.
- Iteratively represent all subdiagrams $(L_\xi \mid \xi < \alpha)$, for $\alpha < \omega_1$, as homomorphic images of $\Bbbk$-vector lattices diagrams ($\Bbbk$ given countable totally ordered division ring).
- The next slide describes what the induction step looks like.

Idea of the proof (cont'd)

- For $I \subseteq J$ countably infinite, $\mathcal{D} \subseteq \Bbbk^{(J)}$ finite, $a \in \Bbbk^{(J)}$, and a completely normal bounded distributive lattice L, we need to extend a $0,1$-lattice homomorphism $f\colon \operatorname{Op}(\Bbbk^{(I)} \cup \mathcal{D}, \Bbbk^{(J)}) \to L$ to a lattice homomorphism $g\colon \operatorname{Op}(\Bbbk^{(I)} \cup \mathcal{D} \cup \{a\}, \Bbbk^{(J)}) \to L$.

Idea of the proof (cont'd)

- For $I \subseteq J$ countably infinite, $\mathcal{D} \subseteq \Bbbk^{(J)}$ finite, $a \in \Bbbk^{(J)}$, and a completely normal bounded distributive lattice L, we need to extend a $0,1$-lattice homomorphism $f\colon \operatorname{Op}(\Bbbk^{(I)} \cup \mathcal{D}, \Bbbk^{(J)}) \to L$ to a lattice homomorphism $g\colon \operatorname{Op}(\Bbbk^{(I)} \cup \mathcal{D} \cup \{a\}, \Bbbk^{(J)}) \to L$.
- This is done as in the finite case (i.e., extend $f\colon \operatorname{Op}(\mathcal{D}, \Bbbk^{(J)}) \to L$ with $\mathcal{D}$ finite), *via* a more general lattice-theoretical extension lemma.

Idea of the proof (cont'd)

- For $I \subseteq J$ countably infinite, $\mathcal{D} \subseteq \mathbb{k}^{(J)}$ finite, $a \in \mathbb{k}^{(J)}$, and a completely normal bounded distributive lattice L, we need to extend a $0,1$-lattice homomorphism $f\colon \operatorname{Op}(\mathbb{k}^{(I)} \cup \mathcal{D}, \mathbb{k}^{(J)}) \to L$ to a lattice homomorphism $g\colon \operatorname{Op}(\mathbb{k}^{(I)} \cup \mathcal{D} \cup \{a\}, \mathbb{k}^{(J)}) \to L$.
- This is done as in the finite case (i.e., extend $f\colon \operatorname{Op}(\mathcal{D}, \mathbb{k}^{(J)}) \to L$ with $\mathcal{D}$ finite), *via* a more general lattice-theoretical extension lemma. A key point is that the Boolean algebra $\operatorname{Bool}(\mathbb{k}^{(I)} \cup \mathcal{D}, \mathbb{k}^{(J)})$ (generated by $\operatorname{Op}(\mathbb{k}^{(I)} \cup \mathcal{D}, \mathbb{k}^{(J)})$) is relatively complete in $\operatorname{Bool}(\mathbb{k}^{(J)}, \mathbb{k}^{(J)})$.
- There is no longer any need to consider the closure step.

A few references (logic, category theory)

1. J. Adámek and J. Rosický, *Locally Presentable and Accessible Categories*, London Mathematical Society Lecture Notes Series **189**, Cambridge University Press, Cambridge, 1994.
2. P. Gillibert and F. Wehrung, *From Objects to Diagrams for Ranges of Functors*, Springer Lecture Notes **2029**, Springer, Heidelberg, 2011.
3. H.M. Tuuri, *Relative separation theorems for* $\mathscr{L}_{\kappa+\kappa}$, Notre Dame J. Formal Logic **33** (1992), no. 3, 383–401.
4. F. Wehrung, *From non-commutative diagrams to anti-elementary classes*, J. Math. Logic **21**, no. 2 (2021), 2150011.
5. F. Wehrung, *Projective classes as images of accessible functors*, J. Logic Comput., to appear, 45 p.

A few references (spectra)

1. F. Wehrung, *Spectral spaces of countable Abelian lattice-ordered groups*, Trans. Amer. Math. Soc. **371** (2019), no. 3, 2133–2158.
2. F. Wehrung, *Cevian operations on distributive lattices*, J. Pure Appl. Algebra **224** (2020), no. 4, 106202, 23 p.
3. F. Wehrung, *Real spectra and ℓ-spectra of algebras and vector lattices over countable fields*, J. Pure Appl. Algebra **226** (2022), no. 4, Paper No. 106861, 25 p.
4. F. Wehrung, *Real spectrum versus ℓ-spectrum via Brumfiel spectrum*, Algebr. Represent. Theory, to appear, 22 p.
5. M. Ploščica and F. Wehrung, *Spectral subspaces of spectra of Abelian lattice-ordered groups in size aleph one*, preprint, in preparation.

Thanks for your attention!