Sublattices of associahedra and permutohedra

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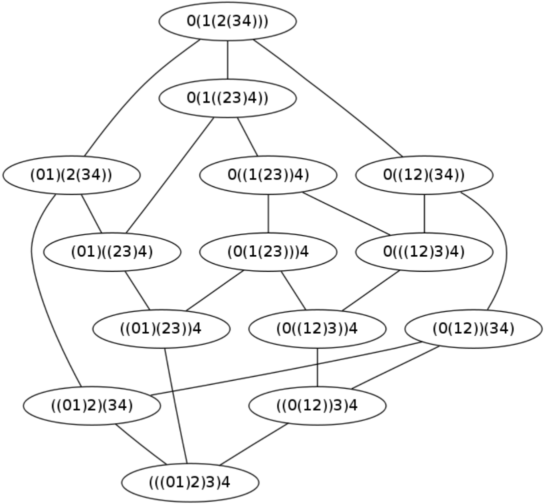
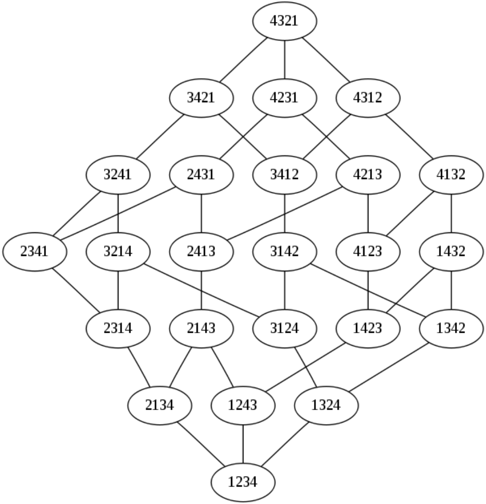
A(4): the associahedron on $4+1$ letters



P(4): the permutohedron on 4 letters



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Theorem (Guilbaud and Rosenstiehl 1963)

The poset $\mathrm{P}(n)$ is a lattice, for each positive integer n.

P(n) as the lattice of all permutations of [n]

- For $\sigma \in \mathfrak{S}_n$, the inversion set

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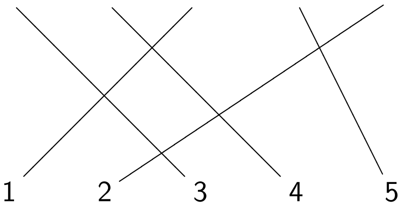

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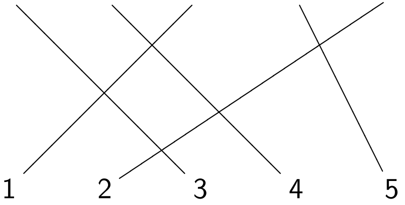

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- Every clopen set has the form $\mathrm{Inv}(\sigma)$, for a (unique) $\sigma \in \mathfrak{S}_n$.

Theorem

$\mathsf{Inv}(\sigma) \subseteq \mathsf{Inv}(\tau)$
if and only if
there is a length-increasing path from σ *to* τ
in the Cayley graph of $\mathfrak{S}_n$.

Associahedra as retracts of permutohedra

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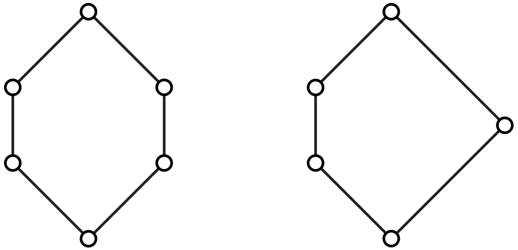
Theorem (mostly Björner and Wachs 1997)

$\mathrm{A}(n)$ is a lattice-theoretical retract of $\mathrm{P}(n)$.

P(3) and A(3)



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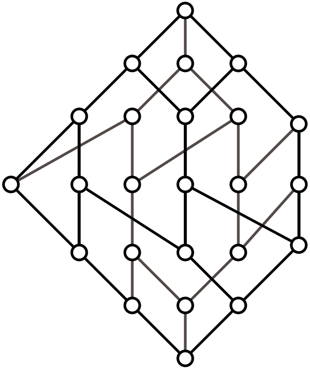

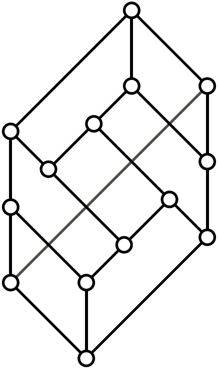

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- Still unknown whether

$$\{\, L \mid \exists n \text{ s.t. } L \hookrightarrow \mathrm{A}(n) \,\}$$

is decidable.

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- Join-dependency relation $\mathbf{D}$: for $p, q \in \mathrm{Ji}(L)$ and $p \neq q$,

$$p \,\mathbf{D}\, q \quad \text{if} \quad \exists x \text{ s.t. } p \leq q \vee x \text{ and } p \nleq q_* \vee x\,.$$

  L is *lower bounded* if $\mathbf{D}$ has no cycle.
  L is *bounded* if L and L^{op} are both lower bounded.

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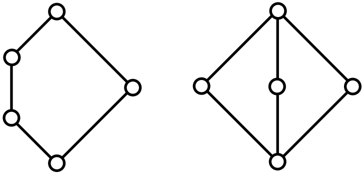

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- Caspard's result was extended to all finite Coxeter lattices by Caspard, Le Conte de Poly-Barbut, and Morvan (2004).
- It follows that every quotient of a sublattice of a permutohedron (associahedron) is bounded.

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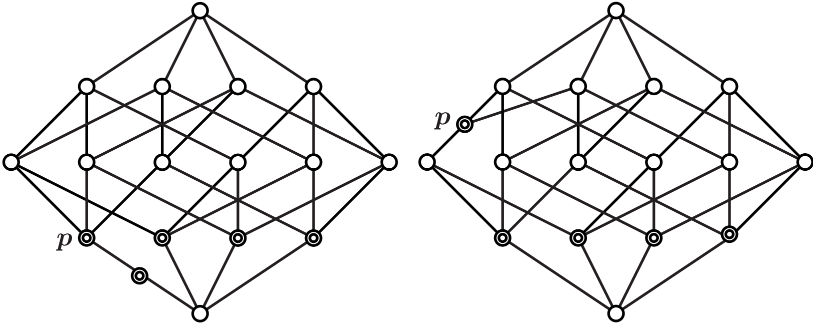
- Conjecture easy to verify for finite distributive lattices.
- Strangely, a similar conjecture for permutohedra was not stated at that time.

The lattices B(m, n)



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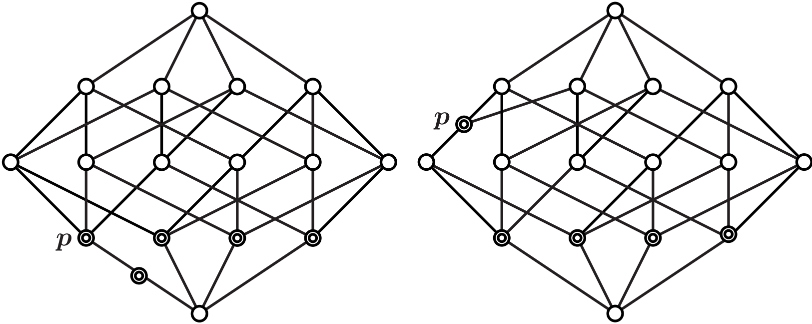

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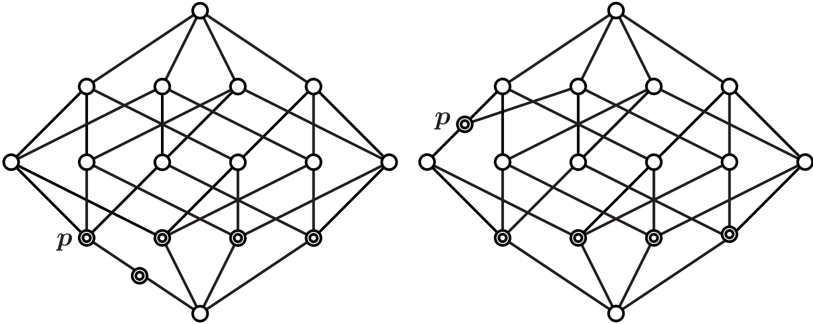

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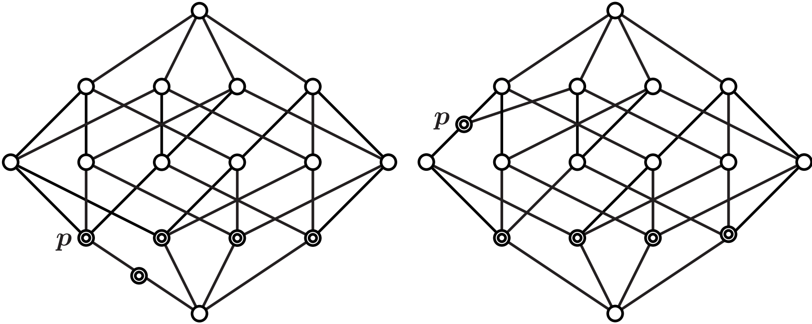

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- All lattices B(m, n) are bounded.
- The lattices B(m, n) and B(n, m) are opposite ("dual").

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In particular:

neither B(2, 2) nor P(4) can be embedded into any A(n).

Polarized measures: duality for finite lattices at work

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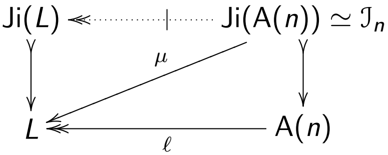

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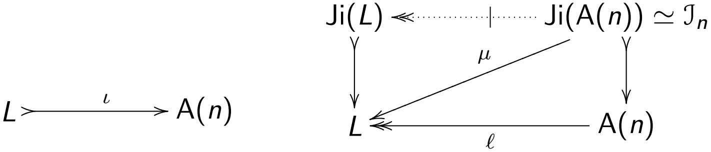

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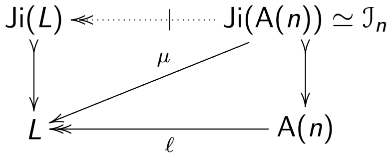

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2. $\mu(i,k) \leq \mu(i,j) \vee \mu(j,k)$,
3. $\mu(i,j) \leq \mathbf{a} \vee \mathbf{b}$ implies

$$i = z_0 < z_1 < \cdots < z_m = j \quad \text{and}$$

$$\text{either } \mu(z_i, z_{i+1}) \leq \mathbf{a} \text{ or } \mu(z_i, z_{i+1}) \leq \mathbf{b},$$

for each $i < m$.

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- The Gazpacho identity (Veg_2):

$$(a_1 \vee b_1) \wedge (a_2 \vee b_2) \leq \bigvee_{i=1}^{2} \bigwedge_{j=1}^{2} (a_i \vee \tilde{b}_j),$$

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is satisfied by all $A(n)$ but not by $P(4)$.

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- For a finite lattice L, certain U-polarized measures with values in L correspond to lattice embeddings of L into certain subdirectly irreducible quotients $\mathrm{P}_U(n)$ of $\mathrm{P}(n)$ (see next page).

Cambrian lattices of type A

- For $U \subseteq [n]$, say that $\mathbf{a} \subseteq \mathcal{I}_n$ is a *U-subset* if

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- $\mathrm{P}(n)$ is a subdirect product of all $\mathrm{P}_U(n)$ for $U \subseteq [n]$.

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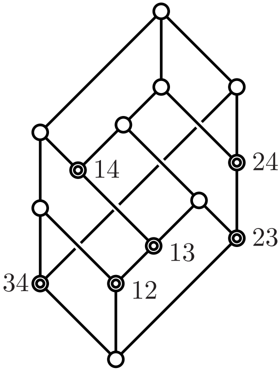

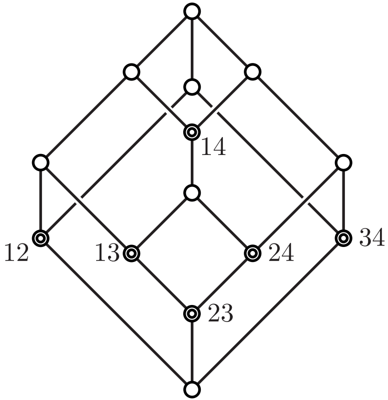

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$B(3,3)$ is a homomorphic image of a sublattice of $P(12)$.

- We prove that a certain $P_U(12)$ does not satisfy the splitting identity of $B(3,3)$:

$$\bigwedge_{1 \le j \le 3} (x_1 \vee x_2 \vee x_3 \vee y_j) \le \bigvee_{1 \le i \le 3} (\hat{x}_i \wedge \hat{y}_1 \wedge \hat{y}_2 \wedge \hat{y}_3)\,,$$

where $x := x_1 \vee x_2 \vee x_3$, $y := y_1 \vee y_2 \vee y_3$,
$\hat{x}_1 := x_2 \vee x_3 \vee y$, $\hat{y}_1 := y_2 \vee y_3 \vee x$, *etc.*

No separating identity for B(3, 3) (cont'd)

- Relevant values of the x_i, y_i obtained with help of the Prover9 -Mace4 program (yields $U = \{5, 6, 9, 10, 11\}$).

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- Verified above in the case of B(3, 3) (with P(12)).