Nonstable K-theory of regular rings and Banaschewski functions

Friedrich Wehrung

Université de Caen
LMNO, UMR 6139
Département de Mathématiques
14032 Caen cedex
E-mail: wehrung@math.unicaen.fr
URL: http://www.math.unicaen.fr/˜wehrung

August 9-13, 2010

$\mathrm{FP}(R)$ and $\mathbb{V}(R)$

- For a unital (associative) ring R, set

$$\begin{aligned}\mathrm{FP}(R) &:= \{X \text{ right } R\text{-module} \mid X \text{ fin. gen. projective}\}\\ &= \{X \mid (\exists n)(\exists Y)(X \oplus Y = R_R^n)\}\,.\end{aligned}$$

FP(R) and $\mathbb{V}(R)$

- For a unital (associative) ring R, set

$$\begin{aligned}\mathrm{FP}(R) &:= \{X \text{ right } R\text{-module} \mid X \text{ fin. gen. projective}\} \\ &= \{X \mid (\exists n)(\exists Y)(X \oplus Y = R_R^n)\}\,.\end{aligned}$$

- For $X \in \mathrm{FP}(R)$, set $[X] :=$ isomorphism class of X.

$\mathrm{FP}(R)$ and $\mathbb{V}(R)$

- For a unital (associative) ring R, set

$$\begin{aligned}\mathrm{FP}(R) &:= \{X \text{ right } R\text{-module} \mid X \text{ fin. gen. projective}\}\\ &= \{X \mid (\exists n)(\exists Y)(X \oplus Y = R_R^n)\}\,.\end{aligned}$$

- For $X \in \mathrm{FP}(R)$, set $[X] :=$ isomorphism class of X.
- Then define $[X] + [Y] := [X \oplus Y]$.

FP(R) and $\mathbb{V}(R)$

- For a unital (associative) ring R, set

$$\begin{aligned}\mathrm{FP}(R) &:= \{X \text{ right } R\text{-module} \mid X \text{ fin. gen. projective}\}\\ &= \{X \mid (\exists n)(\exists Y)(X \oplus Y = R_R^n)\}\,.\end{aligned}$$

- For $X \in \mathrm{FP}(R)$, set $[X] :=$ isomorphism class of X.
- Then define $[X] + [Y] := [X \oplus Y]$.
- $\mathbb{V}(R) := \{[X] \mid X \in \mathrm{FP}(R)\}$, endowed with addition, is a commutative monoid (encodes the nonstable K-theory of R).

$\mathrm{FP}(R)$ and $\mathbb{V}(R)$

- For a unital (associative) ring R, set

$$\begin{aligned}\mathrm{FP}(R) &:= \{X \text{ right } R\text{-module} \mid X \text{ fin. gen. projective}\}\\ &= \{X \mid (\exists n)(\exists Y)(X \oplus Y = R_R^n)\}\,.\end{aligned}$$

- For $X \in \mathrm{FP}(R)$, set $[X] :=$ isomorphism class of X.
- Then define $[X] + [Y] := [X \oplus Y]$.
- $\mathbb{V}(R) := \{[X] \mid X \in \mathrm{FP}(R)\}$, endowed with addition, is a commutative monoid (encodes the nonstable K-theory of R). It is conical: $\alpha + \beta = 0 \Rightarrow \alpha = \beta = 0$.

FP(R) and $\mathbb{V}(R)$

- For a unital (associative) ring R, set

$$\begin{aligned}\mathrm{FP}(R) &:= \{X \text{ right } R\text{-module} \mid X \text{ fin. gen. projective}\} \\ &= \{X \mid (\exists n)(\exists Y)(X \oplus Y = R_R^n)\}\,.\end{aligned}$$

- For $X \in \mathrm{FP}(R)$, set $[X] :=$ isomorphism class of X.
- Then define $[X] + [Y] := [X \oplus Y]$.
- $\mathbb{V}(R) := \{[X] \mid X \in \mathrm{FP}(R)\}$, endowed with addition, is a commutative monoid (encodes the nonstable K-theory of R). It is conical: $\alpha + \beta = 0 \Rightarrow \alpha = \beta = 0$. The element $[R]$ is an order-unit: $\forall \alpha\ \exists \beta\ \exists n\ \alpha + \beta = n[R]$.

FP(R) and $\mathbb{V}(R)$

- For a unital (associative) ring R, set

$$\begin{aligned} \mathrm{FP}(R) &:= \{X \text{ right } R\text{-module} \mid X \text{ fin. gen. projective}\} \\ &= \{X \mid (\exists n)(\exists Y)(X \oplus Y = R_R^n)\}\,. \end{aligned}$$

- For $X \in \mathrm{FP}(R)$, set $[X] :=$ isomorphism class of X.
- Then define $[X] + [Y] := [X \oplus Y]$.
- $\mathbb{V}(R) := \{[X] \mid X \in \mathrm{FP}(R)\}$, endowed with addition, is a commutative monoid (encodes the nonstable K-theory of R). It is conical: $\alpha + \beta = 0 \Rightarrow \alpha = \beta = 0$. The element $[R]$ is an order-unit: $\forall \alpha\ \exists \beta\ \exists n\ \alpha + \beta = n[R]$.
- The definition is left-right symmetric.

FP(R) and $\mathbb{V}(R)$

- For a unital (associative) ring R, set

$$\begin{aligned}\mathrm{FP}(R) &:= \{X \text{ right } R\text{-module} \mid X \text{ fin. gen. projective}\}\\ &= \{X \mid (\exists n)(\exists Y)(X \oplus Y = R_R^n)\}\,.\end{aligned}$$

- For $X \in \mathrm{FP}(R)$, set $[X] :=$ isomorphism class of X.
- Then define $[X] + [Y] := [X \oplus Y]$.
- $\mathbb{V}(R) := \{[X] \mid X \in \mathrm{FP}(R)\}$, endowed with addition, is a commutative monoid (encodes the nonstable K-theory of R). It is conical: $\alpha + \beta = 0 \Rightarrow \alpha = \beta = 0$. The element $[R]$ is an order-unit: $\forall\alpha\ \exists\beta\ \exists n\ \alpha + \beta = n[R]$.
- The definition is left-right symmetric.
- $\mathbb{V}(R) \cong \mathbb{Z}^+ = \{0, 1, 2, \dots\}$ if R is a division ring.

What can $\mathbb{V}(R)$ be?

- On a commutative monoid M, $x \leq y :\Leftrightarrow (\exists z)(x + z = y)$; algebraic preordering of M.

What can $\mathbb{V}(R)$ be?

- On a commutative monoid M, $x \leq y :\Leftrightarrow (\exists z)(x + z = y)$; algebraic preordering of M.
- order-unit of M: any $e \in M$ such that $(\forall x \in M)(\exists n \in \mathbb{N})(x \leq ne)$.

What can $\mathbb{V}(R)$ be?

- On a commutative monoid M, $x \leq y :\Leftrightarrow (\exists z)(x + z = y)$; algebraic preordering of M.
- order-unit of M: any $e \in M$ such that $(\forall x \in M)(\exists n \in \mathbb{N})(x \leq ne)$.
- Every conical commutative monoid with order-unit is isomorphic to $\mathbb{V}(R)$, for some hereditary, unital ring R (Bergman 1974 in the finitely generated case, Bergman and Dicks 1978 in the general case).

The realization problem in the regular case

Fundamental problem (Goodearl 1995)

The realization problem in the regular case

Fundamental problem (Goodearl 1995)

Which monoids are representable, that is, appear as $\mathbb{V}(R)$ for a (von Neumann) regular ring R?

The realization problem in the regular case

Fundamental problem (Goodearl 1995)

Which monoids are representable, that is, appear as $\mathbb{V}(R)$ for a (von Neumann) regular ring R?

- A survey paper about this problem: P. Ara, *The realization problem for von Neumann regular rings*. Ring theory 2007, 21–37, World Sci. Publ., Hackensack, NJ, 2009 (also arXiv:0802.1872).

The realization problem in the regular case

Fundamental problem (Goodearl 1995)

Which monoids are representable, that is, appear as $\mathbb{V}(R)$ for a (von Neumann) regular ring R?

- A survey paper about this problem: P. Ara, *The realization problem for von Neumann regular rings*. Ring theory 2007, 21–37, World Sci. Publ., Hackensack, NJ, 2009 (also arXiv:0802.1872).
- "Conicality" $(\forall x, y)(x + y = 0 \Rightarrow x = y = 0)$ and "existence of an order-unit" $(\forall x)(\exists n)(x \leq ne)$ not sufficient. Another condition, whose necessity was proved by Goodearl and Handelman (1975), is

The realization problem in the regular case

Fundamental problem (Goodearl 1995)

Which monoids are representable, that is, appear as $\mathbb{V}(R)$ for a (von Neumann) regular ring R?

- A survey paper about this problem: P. Ara, *The realization problem for von Neumann regular rings*. Ring theory 2007, 21–37, World Sci. Publ., Hackensack, NJ, 2009 (also arXiv:0802.1872).
- "Conicality" $(\forall x, y)(x + y = 0 \Rightarrow x = y = 0)$ and "existence of an order-unit" $(\forall x)(\exists n)(x \leq ne)$ not sufficient. Another condition, whose necessity was proved by Goodearl and Handelman (1975), is
- The refinement condition: $a_0 + a_1 = b_0 + b_1 \Rightarrow$ there are $c_{i,j}$ $(i, j \in \{0, 1\})$ such that $a_i = c_{i,0} + c_{i,1}$ and $b_i = c_{0,i} + c_{1,i}$ $\forall i < 2$.

Variants of the problem

Unrestricted Realization Problem

Variants of the problem

Unrestricted Realization Problem

Is every conical refinement monoid with order-unit representable?

Variants of the problem

Unrestricted Realization Problem

Is every conical refinement monoid with order-unit representable?

Definition

Variants of the problem

Unrestricted Realization Problem

Is every conical refinement monoid with order-unit representable?

Definition

A dimension group is a partially ordered abelian group G which is directed ($\forall x, y$, $\exists z$ such that $x \leq z$ and $y \leq z$), unperforated ($\forall m \in \mathbb{N}$, $\forall x$, $mx \geq 0 \Rightarrow x \geq 0$), and such that $G^+ := \{x \in G \mid x \geq 0\}$ is a refinement monoid.

Variants of the problem

Unrestricted Realization Problem

Is every conical refinement monoid with order-unit representable?

Definition

A dimension group is a partially ordered abelian group G which is directed ($\forall x, y$, $\exists z$ such that $x \leq z$ and $y \leq z$), unperforated ($\forall m \in \mathbb{N}$, $\forall x$, $mx \geq 0 \Rightarrow x \geq 0$), and such that $G^+ := \{x \in G \mid x \geq 0\}$ is a refinement monoid.

- Dimension groups are exactly the direct limits of (componentwise ordered) $\mathbb{Z}^n$ with positive homomorphisms (Effros, Handelman, and Shen 1980; equivalent semigroup statement due to Grillet in 1976).

Unrestricted Realization Problem (cont'd)

- By combining this with a 1976 result by Elliott, it follows that G^+ is representable, for any countable dimension group G with order-unit.

Unrestricted Realization Problem (cont'd)

- By combining this with a 1976 result by Elliott, it follows that G^+ is representable, for any countable dimension group G with order-unit.
- Extended to dimension groups of cardinality $\aleph_1$ by Goodearl and Handelman (1986).

Unrestricted Realization Problem (cont'd)

- By combining this with a 1976 result by Elliott, it follows that G^+ is representable, for any countable dimension group G with order-unit.
- Extended to dimension groups of cardinality $\aleph_1$ by Goodearl and Handelman (1986). In both cases, the representing ring can be taken locally matricial (over any given field).

Unrestricted Realization Problem (cont'd)

- By combining this with a 1976 result by Elliott, it follows that G^+ is representable, for any countable dimension group G with order-unit.
- Extended to dimension groups of cardinality $\aleph_1$ by Goodearl and Handelman (1986). In both cases, the representing ring can be taken locally matricial (over any given field).
- Does not extend to dimension groups of cardinality $\geq \aleph_2$ (W. 1998).

Unrestricted Realization Problem (cont'd)

- By combining this with a 1976 result by Elliott, it follows that G^+ is representable, for any countable dimension group G with order-unit.
- Extended to dimension groups of cardinality $\aleph_1$ by Goodearl and Handelman (1986). In both cases, the representing ring can be taken locally matricial (over any given field).
- Does not extend to dimension groups of cardinality $\geq \aleph_2$ (W. 1998). (Situation still mysterious, on that front, for C*-algebras.)

Unrestricted Realization Problem (cont'd)

- By combining this with a 1976 result by Elliott, it follows that G^+ is representable, for any countable dimension group G with order-unit.
- Extended to dimension groups of cardinality $\aleph_1$ by Goodearl and Handelman (1986). In both cases, the representing ring can be taken locally matricial (over any given field).
- Does not extend to dimension groups of cardinality $\geq \aleph_2$ (W. 1998). (Situation still mysterious, on that front, for C*-algebras.)
- Hence the answer to the Unrestricted Realization Problem (for regular rings) is "no".

The realization problem

The Realization Problem

The realization problem

The Realization Problem

Is every (at most) countable conical refinement monoid representable?

The realization problem

The Realization Problem

Is every (at most) countable conical refinement monoid representable?

- This is even open for monoids with $\leq \aleph_1$ elements.

The realization problem

The Realization Problem

Is every (at most) countable conical refinement monoid representable?

- This is even open for monoids with $\leq \aleph_1$ elements.
- If this could hold at $\aleph_1$, then it would also hold at arrows (or even sequences of arrows, or even more...) of countable refinement monoids with order-unit.

The realization problem

The Realization Problem

Is every (at most) countable conical refinement monoid representable?

- This is even open for monoids with $\leq \aleph_1$ elements.
- If this could hold at $\aleph_1$, then it would also hold at arrows (or even sequences of arrows, or even more...) of countable refinement monoids with order-unit. (*Due to categorical work by Gillibert and W. 2010*).
- Special case: the Separativity Conjecture. For finitely generated projective right modules A and B over a regular ring R, does $A_R \oplus B_R \cong A_R^2 \cong B_R^2 \Rightarrow A_R \cong B_R$?

The realization problem

The Realization Problem

Is every (at most) countable conical refinement monoid representable?

- This is even open for monoids with $\leq \aleph_1$ elements.
- If this could hold at $\aleph_1$, then it would also hold at arrows (or even sequences of arrows, or even more...) of countable refinement monoids with order-unit. (*Due to categorical work by Gillibert and W. 2010*).
- Special case: the Separativity Conjecture. For finitely generated projective right modules A and B over a regular ring R, does $A_R \oplus B_R \cong A_R^2 \cong B_R^2 \Rightarrow A_R \cong B_R$?
- The Realization Problem and the Separativity Conjecture contradict each other.

Exchange rings

- A unital ring R is an *exchange ring* if $A = M \oplus N = \bigoplus_{i=1}^{n} A_i$, with $M \in \mathrm{FP}(R)$, implies that $A = M \oplus \bigoplus_{i=1}^{n} A'_i$ for submodules $A'_i \subseteq A_i$.

Exchange rings

- A unital ring R is an *exchange ring* if $A = M \oplus N = \bigoplus_{i=1}^{n} A_i$, with $M \in \mathrm{FP}(R)$, implies that $A = M \oplus \bigoplus_{i=1}^{n} A'_i$ for submodules $A'_i \subseteq A_i$.
- Equivalently (Goodearl + Warfield, Nicholson), $\forall a \in R$, $\exists e \in R$ idempotent, $e \in aR$ and $1 - e \in (1 - a)R$.

Exchange rings

- A unital ring R is an *exchange ring* if $A = M \oplus N = \bigoplus_{i=1}^{n} A_i$, with $M \in \mathrm{FP}(R)$, implies that $A = M \oplus \bigoplus_{i=1}^{n} A_i'$ for submodules $A_i' \subseteq A_i$.
- Equivalently (Goodearl + Warfield, Nicholson), $\forall a \in R$, $\exists e \in R$ idempotent, $e \in aR$ and $1 - e \in (1 - a)R$.
- Every regular ring is an exchange ring (converse false).

Exchange rings

- A unital ring R is an *exchange ring* if $A = M \oplus N = \bigoplus_{i=1}^{n} A_i$, with $M \in \mathrm{FP}(R)$, implies that $A = M \oplus \bigoplus_{i=1}^{n} A'_i$ for submodules $A'_i \subseteq A_i$.
- Equivalently (Goodearl + Warfield, Nicholson), $\forall a \in R$, $\exists e \in R$ idempotent, $e \in aR$ and $1 - e \in (1 - a)R$.
- Every regular ring is an exchange ring (converse false).
- A C*-algebra is an exchange ring iff it has real rank zero.

Exchange rings

- A unital ring R is an *exchange ring* if $A = M \oplus N = \bigoplus_{i=1}^{n} A_i$, with $M \in \mathrm{FP}(R)$, implies that $A = M \oplus \bigoplus_{i=1}^{n} A'_i$ for submodules $A'_i \subseteq A_i$.
- Equivalently (Goodearl + Warfield, Nicholson), $\forall a \in R$, $\exists e \in R$ idempotent, $e \in aR$ and $1 - e \in (1 - a)R$.
- Every regular ring is an exchange ring (converse false).
- A C*-algebra is an exchange ring iff it has real rank zero.
- Both Realization and Separativity are also unsettled for exchange rings.

Dimension groups

(*Dimension group*: directed, unperforated partially ordered abelian group whose positive cone has refinement; equivalently, direct limit of $\mathbb{Z}^n$s.)

Dimension groups

(*Dimension group*: directed, unperforated partially ordered abelian group whose positive cone has refinement; equivalently, direct limit of $\mathbb{Z}^n$s.) Recall the following (Effros, Elliott, Goodearl, Grillet, Handelman, Shen, W.):

Dimension groups

(*Dimension group*: directed, unperforated partially ordered abelian group whose positive cone has refinement; equivalently, direct limit of $\mathbb{Z}^n$s.) Recall the following (Effros, Elliott, Goodearl, Grillet, Handelman, Shen, W.):

Theorem

Dimension groups

(*Dimension group*: directed, unperforated partially ordered abelian group whose positive cone has refinement; equivalently, direct limit of $\mathbb{Z}^n$s.) Recall the following (Effros, Elliott, Goodearl, Grillet, Handelman, Shen, W.):

Theorem

The positive cone of any dimension group with order-unit with $\leq \aleph_1$ elements is representable. For cardinalities $\geq \aleph_2$, there are counterexamples.

Dimension groups

(*Dimension group*: directed, unperforated partially ordered abelian group whose positive cone has refinement; equivalently, direct limit of $\mathbb{Z}^n$s.) Recall the following (Effros, Elliott, Goodearl, Grillet, Handelman, Shen, W.):

Theorem

The positive cone of any dimension group with order-unit with $\leq \aleph_1$ elements is representable. For cardinalities $\geq \aleph_2$, there are counterexamples.

The representation problem is open even for general countable, cancellative refinement monoids (= positive cones of interpolation groups).

Distributive semilattices

- A semilattice is a monoid $(M, +, 0)$ such that $x + x = x$ for each $x \in M$.

Distributive semilattices

- A semilattice is a monoid $(M, +, 0)$ such that $x + x = x$ for each $x \in M$.
- Algebraic preordering: $x \leq y \Leftrightarrow x + y = y$.

Distributive semilattices

- A semilattice is a monoid $(M, +, 0)$ such that $x + x = x$ for each $x \in M$.
- Algebraic preordering: $x \leq y \Leftrightarrow x + y = y$.
- A semilattice has refinement iff it is distributive, that is,

$$(\forall a, b, c)(c \leq a + b \Rightarrow (\exists x \leq a)(\exists y \leq b)(c = x + y)).$$

Distributive semilattices

- A semilattice is a monoid $(M, +, 0)$ such that $x + x = x$ for each $x \in M$.
- Algebraic preordering: $x \leq y \Leftrightarrow x + y = y$.
- A semilattice has refinement iff it is distributive, that is,

$$(\forall a, b, c)(c \leq a + b \Rightarrow (\exists x \leq a)(\exists y \leq b)(c = x + y)).$$

Theorem (W. 2000)

Distributive semilattices

- A semilattice is a monoid $(M, +, 0)$ such that $x + x = x$ for each $x \in M$.
- Algebraic preordering: $x \leq y \Leftrightarrow x + y = y$.
- A semilattice has refinement iff it is distributive, that is,

$$(\forall a, b, c)(c \leq a + b \Rightarrow (\exists x \leq a)(\exists y \leq b)(c = x + y)) .$$

Theorem (W. 2000)

Every distributive semilattice with $\leq \aleph_1$ elements is representable. The bound $\aleph_1$ is optimal.

Distributive semilattices

- A semilattice is a monoid $(M, +, 0)$ such that $x + x = x$ for each $x \in M$.
- Algebraic preordering: $x \leq y \Leftrightarrow x + y = y$.
- A semilattice has refinement iff it is distributive, that is,

$$(\forall a, b, c)(c \leq a + b \Rightarrow (\exists x \leq a)(\exists y \leq b)(c = x + y)) \,.$$

Theorem (W. 2000)

Every distributive semilattice with $\leq \aleph_1$ elements is representable. The bound $\aleph_1$ is optimal.

Involves the natural extension of $\mathbb{V}(R)$ to the non-unital case.

Graph monoids

- A quiver is a quadruple $E = (E^0, E^1, s, t)$, where both E^0 and E^1 are sets and $s, t \colon E^1 \to E^0$. The set E^0 is the vertex set, E^1 is the edge set, s is the source map, and t is the target map.

Graph monoids

- A quiver is a quadruple $E = (E^0, E^1, s, t)$, where both E^0 and E^1 are sets and $s, t \colon E^1 \to E^0$. The set E^0 is the vertex set, E^1 is the edge set, s is the source map, and t is the target map.
- We say that E is row-finite if $s^{-1}\{v\}$ is finite $\forall v \in E^0$.

Graph monoids

- A quiver is a quadruple $E = (E^0, E^1, s, t)$, where both E^0 and E^1 are sets and $s, t\colon E^1 \to E^0$. The set E^0 is the vertex set, E^1 is the edge set, s is the source map, and t is the target map.
- We say that E is row-finite if $s^{-1}\{v\}$ is finite $\forall v \in E^0$.
- The graph monoid of E, denoted by $\mathrm{M}(E)$, is the commutative monoid defined by generators $\overline{v}$, for $v \in E^0$, and relations

$$\overline{v} = \sum\left(\overline{t(e)} \mid e \in s^{-1}\{v\}\right),$$

for each $v \in E^0$ such that $s^{-1}\{v\} \neq \varnothing$.

Graph monoids

- A quiver is a quadruple $E = (E^0, E^1, s, t)$, where both E^0 and E^1 are sets and $s, t \colon E^1 \to E^0$. The set E^0 is the vertex set, E^1 is the edge set, s is the source map, and t is the target map.
- We say that E is row-finite if $s^{-1}\{v\}$ is finite $\forall v \in E^0$.
- The graph monoid of E, denoted by $\mathrm{M}(E)$, is the commutative monoid defined by generators $\overline{v}$, for $v \in E^0$, and relations

$$\overline{v} = \sum \left(\overline{t(e)} \mid e \in s^{-1}\{v\}\right),$$

for each $v \in E^0$ such that $s^{-1}\{v\} \neq \varnothing$.

Theorem (Ara, Moreno, and Pardo 2007)

Graph monoids

- A quiver is a quadruple $E = (E^0, E^1, s, t)$, where both E^0 and E^1 are sets and $s, t\colon E^1 \to E^0$. The set E^0 is the vertex set, E^1 is the edge set, s is the source map, and t is the target map.
- We say that E is row-finite if $s^{-1}\{v\}$ is finite $\forall v \in E^0$.
- The graph monoid of E, denoted by $\mathrm{M}(E)$, is the commutative monoid defined by generators $\overline{v}$, for $v \in E^0$, and relations

$$\overline{v} = \sum \left(\overline{t(e)} \mid e \in s^{-1}\{v\}\right),$$

for each $v \in E^0$ such that $s^{-1}\{v\} \neq \varnothing$.

Theorem (Ara, Moreno, and Pardo 2007)

$\mathrm{M}(E)$ is a conical refinement monoid, for every row-finite quiver E.

Realization of graph monoids

Theorem (Ara and Brustenga 2007)

Realization of graph monoids

Theorem (Ara and Brustenga 2007)

The graph monoid $\mathrm{M}(E)$ is representable, for every row-finite quiver E.

Realization of graph monoids

Theorem (Ara and Brustenga 2007)

The graph monoid $\mathrm{M}(E)$ is representable, for every row-finite quiver E.

- Involves, again, the natural extension of $\mathbb{V}(R)$ to the non-unital case.

Realization of graph monoids

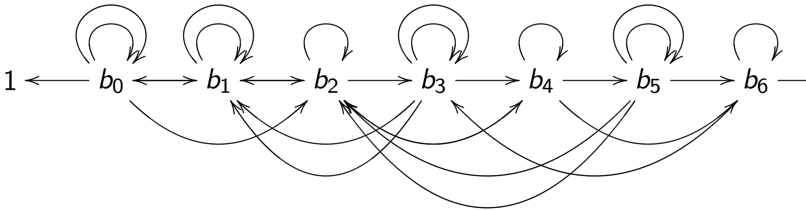
Theorem (Ara and Brustenga 2007)

The graph monoid $\mathrm{M}(E)$ is representable, for every row-finite quiver E.

- Involves, again, the natural extension of $\mathbb{V}(R)$ to the non-unital case.
- Ara and Brustenga construct, for any field K, a regular K-algebra $\mathrm{Q}_K(E)$ such that $\mathrm{M}(E) \cong \mathbb{V}(\mathrm{Q}_K(E))$.

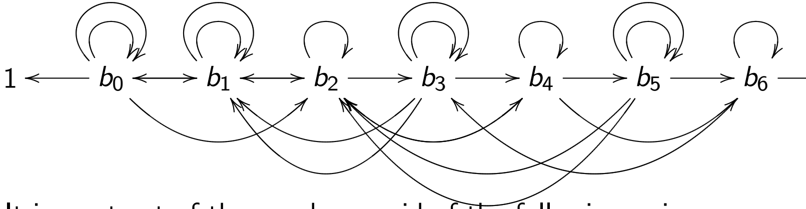
A strange quiver

The monoid $\mathbb{Z}^\infty := \{0, 1, 2, \dots\} \cup \{\infty\}$ can be represented by the following infinite, row-finite quiver (Ara, Perera, and W. 2008):

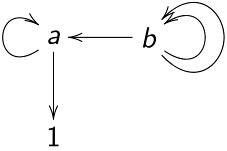


A strange quiver

The monoid $\mathbb{Z}^{\infty} := \{0, 1, 2, \ldots\} \cup \{\infty\}$ can be represented by the following infinite, row-finite quiver (Ara, Perera, and W. 2008):



It is a retract of the graph monoid of the following quiver:



More graph monoids

- Graph monoids are quite special refinement monoids. In particular, $\mathrm{M}(E)$ is always separative ($2x = 2y = x + y \Rightarrow x = y$). In fact, if E is finite, then $\mathrm{M}(E)$ is primely generated.

More graph monoids

- Graph monoids are quite special refinement monoids. In particular, $\mathrm{M}(E)$ is always separative ($2x = 2y = x + y \Rightarrow x = y$). In fact, if E is finite, then $\mathrm{M}(E)$ is primely generated.
- Not every primely generated refinement monoid is a graph monoid. Easiest example (Ara, Perera, and W. 2008): $p = p + a = p + b$.

More graph monoids

- Graph monoids are quite special refinement monoids. In particular, $\mathrm{M}(E)$ is always separative ($2x = 2y = x + y \Rightarrow x = y$). In fact, if E is finite, then $\mathrm{M}(E)$ is primely generated.
- Not every primely generated refinement monoid is a graph monoid. Easiest example (Ara, Perera, and W. 2008): $p = p + a = p + b$.
- This example is representable, due to the following result.

More graph monoids

- Graph monoids are quite special refinement monoids. In particular, $\mathrm{M}(E)$ is always separative ($2x = 2y = x + y \Rightarrow x = y$). In fact, if E is finite, then $\mathrm{M}(E)$ is primely generated.
- Not every primely generated refinement monoid is a graph monoid. Easiest example (Ara, Perera, and W. 2008): $p = p + a = p + b$.
- This example is representable, due to the following result.

Theorem (Ara 2010)

More graph monoids

- Graph monoids are quite special refinement monoids. In particular, $\mathrm{M}(E)$ is always separative ($2x = 2y = x + y \Rightarrow x = y$). In fact, if E is finite, then $\mathrm{M}(E)$ is primely generated.
- Not every primely generated refinement monoid is a graph monoid. Easiest example (Ara, Perera, and W. 2008): $p = p + a = p + b$.
- This example is representable, due to the following result.

Theorem (Ara 2010)

Let P be a finite poset. Then the commutative monoid with generators $\overline{p}$, for $p \in P$, and relations $\overline{q} = \overline{p} + \overline{q}$, for $p < q$ in P, is representable.

More graph monoids

- Graph monoids are quite special refinement monoids. In particular, $\mathrm{M}(E)$ is always separative ($2x = 2y = x + y \Rightarrow x = y$). In fact, if E is finite, then $\mathrm{M}(E)$ is primely generated.
- Not every primely generated refinement monoid is a graph monoid. Easiest example (Ara, Perera, and W. 2008): $p = p + a = p + b$.
- This example is representable, due to the following result.

Theorem (Ara 2010)

Let P be a finite poset. Then the commutative monoid with generators $\overline{p}$, for $p \in P$, and relations $\overline{q} = \overline{p} + \overline{q}$, for $p < q$ in P, is representable.

Again, the representing ring can be taken a regular K-algebra, for any given field K.

Continuous dimension scales

- For any ordinal γ, endow

$$\begin{aligned}
\mathbb{Z}_\gamma &:= \mathbb{Z}^+ \cup \{\aleph_\alpha \mid \alpha \leq \gamma\}\,,\\
\mathbb{R}_\gamma &:= \mathbb{R}^+ \cup \{\aleph_\alpha \mid \alpha \leq \gamma\}\,,\\
\mathbf{2}_\gamma &:= \{0\} \cup \{\aleph_\alpha \mid \alpha \leq \gamma\}\,.
\end{aligned}$$

with their interval topology.

Continuous dimension scales

- For any ordinal γ, endow

$$\mathbb{Z}_\gamma := \mathbb{Z}^+ \cup \{\aleph_\alpha \mid \alpha \leq \gamma\}\,,$$
$$\mathbb{R}_\gamma := \mathbb{R}^+ \cup \{\aleph_\alpha \mid \alpha \leq \gamma\}\,,$$
$$\mathbf{2}_\gamma := \{0\} \cup \{\aleph_\alpha \mid \alpha \leq \gamma\}\,.$$

with their interval topology.

Definition (Goodearl and W. 2005)

Continuous dimension scales

- For any ordinal γ, endow

$$\mathbb{Z}_\gamma := \mathbb{Z}^+ \cup \{\aleph_\alpha \mid \alpha \leq \gamma\},$$
$$\mathbb{R}_\gamma := \mathbb{R}^+ \cup \{\aleph_\alpha \mid \alpha \leq \gamma\},$$
$$\mathbf{2}_\gamma := \{0\} \cup \{\aleph_\alpha \mid \alpha \leq \gamma\}.$$

with their interval topology.

Definition (Goodearl and W. 2005)

A continuous dimension scale is a monoid that can be represented as a lower subset in a product of the form

$$\mathbf{C}(\Omega_{\mathrm{I}}, \mathbb{Z}_\gamma) \times \mathbf{C}(\Omega_{\mathrm{II}}, \mathbb{R}_\gamma) \times \mathbf{C}(\Omega_{\mathrm{III}}, \mathbf{2}_\gamma),$$

Continuous dimension scales

- For any ordinal γ, endow

$$\mathbb{Z}_\gamma := \mathbb{Z}^+ \cup \{\aleph_\alpha \mid \alpha \leq \gamma\}\,,$$
$$\mathbb{R}_\gamma := \mathbb{R}^+ \cup \{\aleph_\alpha \mid \alpha \leq \gamma\}\,,$$
$$\mathbf{2}_\gamma := \{0\} \cup \{\aleph_\alpha \mid \alpha \leq \gamma\}\,.$$

with their interval topology.

Definition (Goodearl and W. 2005)

A continuous dimension scale is a monoid that can be represented as a lower subset in a product of the form

$$\mathbf{C}(\Omega_{\mathrm{I}}, \mathbb{Z}_\gamma) \times \mathbf{C}(\Omega_{\mathrm{II}}, \mathbb{R}_\gamma) \times \mathbf{C}(\Omega_{\mathrm{III}}, \mathbf{2}_\gamma)\,,$$

where Ω_{I}, Ω_{II}, Ω_{III} are Stone spaces of complete Boolean algebras.

Realizations of continuous dimension scales

Continuous dimension scales can also be characterized by a list of axioms (including conditional completeness for the algebraic ordering, general comparability, etc.).

Realizations of continuous dimension scales

Continuous dimension scales can also be characterized by a list of axioms (including conditional completeness for the algebraic ordering, general comparability, etc.).

Theorem (Goodearl and W. 2005)

Realizations of continuous dimension scales

Continuous dimension scales can also be characterized by a list of axioms (including conditional completeness for the algebraic ordering, general comparability, etc.).

Theorem (Goodearl and W. 2005)

- The monoid $\mathbb{V}(R)$ is a continuous dimension scale, for every right self-injective regular ring R. Every continuous dimension scale can be realized in this way.

Realizations of continuous dimension scales

Continuous dimension scales can also be characterized by a list of axioms (including conditional completeness for the algebraic ordering, general comparability, etc.).

Theorem (Goodearl and W. 2005)

- The monoid $\mathbb{V}(R)$ is a continuous dimension scale, for every right self-injective regular ring R. Every continuous dimension scale can be realized in this way.
- A similar result holds for AW*-algebras.

Realizations of continuous dimension scales

Continuous dimension scales can also be characterized by a list of axioms (including conditional completeness for the algebraic ordering, general comparability, etc.).

Theorem (Goodearl and W. 2005)

- The monoid $\mathbb{V}(R)$ is a continuous dimension scale, for every right self-injective regular ring R. Every continuous dimension scale can be realized in this way.
- A similar result holds for AW*-algebras.
- For W*-algebras, the spaces Ω_i must be hyperstonian (and then there is no further restriction).

Dependence of the field

- In all four classes of representable monoids above (dimension groups; distributive semilattices; graph monoids; continuous dimension scales), the representing ring R can be taken an algebra over any given field.

Dependence of the field

- In all four classes of representable monoids above (dimension groups; distributive semilattices; graph monoids; continuous dimension scales), the representing ring R can be taken an algebra over any given field. Things are not always that nice.

Dependence of the field

- In all four classes of representable monoids above (dimension groups; distributive semilattices; graph monoids; continuous dimension scales), the representing ring R can be taken an algebra over any given field. Things are not always that nice.
- Chuang and Lee published in 1990 an example of a non unit-regular, residually Artinian regular algebra (over a countable field). For any such ring R, there is no regular algebra $\overline{R}$ over an uncountable field such that $\mathbb{V}(R) \cong \mathbb{V}(\overline{R})$ (W. 2007). More generally,

Dependence of the field

- In all four classes of representable monoids above (dimension groups; distributive semilattices; graph monoids; continuous dimension scales), the representing ring R can be taken an algebra over any given field. Things are not always that nice.
- Chuang and Lee published in 1990 an example of a non unit-regular, residually Artinian regular algebra (over a countable field). For any such ring R, there is no regular algebra $\overline{R}$ over an uncountable field such that $\mathbb{V}(R) \cong \mathbb{V}(\overline{R})$ (W. 2007). More generally,

Theorem (Goodearl 2008)

Dependence of the field

- In all four classes of representable monoids above (dimension groups; distributive semilattices; graph monoids; continuous dimension scales), the representing ring R can be taken an algebra over any given field. Things are not always that nice.
- Chuang and Lee published in 1990 an example of a non unit-regular, residually Artinian regular algebra (over a countable field). For any such ring R, there is no regular algebra $\overline{R}$ over an uncountable field such that $\mathbb{V}(R) \cong \mathbb{V}(\overline{R})$ (W. 2007). More generally,

Theorem (Goodearl 2008)

Let M be a conical refinement monoid with an order-unit e and a monoid homomorphism $s\colon M \to \mathbb{R}^+$ such that $s(e) = 1$ and $s^{-1}\{0\} = \{0\}$. If M is not cancellative, then there is no regular algebra R over an uncountable field such that $M \cong \mathbb{V}(R)$.

Banaschewski's result

Denote by Sub V the set of all subspaces of a vector space V (over any division ring), ordered by $\subseteq$.

Banaschewski's result

Denote by $\operatorname{Sub} V$ the set of all subspaces of a vector space V (over any division ring), ordered by $\subseteq$.

Theorem (Banaschewski 1957)

Banaschewski's result

Denote by Sub V the set of all subspaces of a vector space V (over any division ring), ordered by $\subseteq$.

Theorem (Banaschewski 1957)

Let V be a vector space. Then there exists a Banaschewski function on Sub V, that is, a map $f\colon \operatorname{Sub} V \to \operatorname{Sub} V$ such that

Banaschewski's result

Denote by $\operatorname{Sub} V$ the set of all subspaces of a vector space V (over any division ring), ordered by $\subseteq$.

Theorem (Banaschewski 1957)

Let V be a vector space. Then there exists a Banaschewski function on $\operatorname{Sub} V$, that is, a map $f \colon \operatorname{Sub} V \to \operatorname{Sub} V$ such that

- $V = X \oplus f(X)$ for each $X \in \operatorname{Sub} V$.

Banaschewski's result

Denote by $\operatorname{Sub} V$ the set of all subspaces of a vector space V (over any division ring), ordered by $\subseteq$.

Theorem (Banaschewski 1957)

Let V be a vector space. Then there exists a Banaschewski function on $\operatorname{Sub} V$, that is, a map $f\colon \operatorname{Sub} V \to \operatorname{Sub} V$ such that

- $V = X \oplus f(X)$ for each $X \in \operatorname{Sub} V$.
- f is antitone, that is, $X \subseteq Y$ implies that $f(Y) \subseteq f(X)$.

Proof of Banaschewski's Theorem

- Denote by $\lhd$ a strict well-ordering of a basis B of V. We set

Proof of Banaschewski's Theorem

- Denote by $\lhd$ a strict well-ordering of a basis B of V. We set

$$\begin{aligned} \langle X \rangle &:= \text{subspace of } V \text{ generated by } X\,, && \forall X \in \operatorname{Sub} V\,; \\ B \downdownarrows b &:= \{x \in B \mid x \lhd b\}\,, && \forall b \in B\,; \\ F(X) &:= \{b \in B \mid b \notin X + \langle B \downdownarrows b \rangle\}\,, && \forall X \in \operatorname{Sub} V\,; \\ f(X) &:= \langle F(X) \rangle\,, && \forall X \in \operatorname{Sub} V\,. \end{aligned}$$

Proof of Banaschewski's Theorem

- Denote by $\lhd$ a strict well-ordering of a basis B of V. We set

$$\begin{aligned} \langle X \rangle &:= \text{subspace of } V \text{ generated by } X\,, && \forall X \in \operatorname{Sub} V\,;\\ B \downdownarrows b &:= \{x \in B \mid x \lhd b\}\,, && \forall b \in B\,;\\ F(X) &:= \{b \in B \mid b \notin X + \langle B \downdownarrows b \rangle\}\,, && \forall X \in \operatorname{Sub} V\,;\\ f(X) &:= \langle F(X) \rangle\,, && \forall X \in \operatorname{Sub} V\,. \end{aligned}$$

- Then $X \subseteq Y$ obviously implies that $F(Y) \subseteq F(X)$, thus $f(Y) \subseteq f(X)$.

Proof of Banaschewski's Theorem

- Denote by $\lhd$ a strict well-ordering of a basis B of V. We set

$$\begin{aligned} \langle X\rangle &:= \text{subspace of } V \text{ generated by } X\,, && \forall X \in \operatorname{Sub} V\,;\\ B \downdownarrows b &:= \{x \in B \mid x \lhd b\}\,, && \forall b \in B\,;\\ F(X) &:= \{b \in B \mid b \notin X + \langle B \downdownarrows b\rangle\}\,, && \forall X \in \operatorname{Sub} V\,;\\ f(X) &:= \langle F(X)\rangle\,, && \forall X \in \operatorname{Sub} V\,. \end{aligned}$$

- Then $X \subseteq Y$ obviously implies that $F(Y) \subseteq F(X)$, thus $f(Y) \subseteq f(X)$.
- Verify that $X \cap f(X) = \{0\}$ (uses $\lhd$ linear ordering).

Proof of Banaschewski's Theorem

- Denote by $\lhd$ a strict well-ordering of a basis B of V. We set

$$\begin{aligned}
\langle X\rangle &:= \text{subspace of } V \text{ generated by } X\,, && \forall X\in\operatorname{Sub}V\,;\\
B\downdownarrows b &:= \{x\in B\mid x\lhd b\}\,, && \forall b\in B\,;\\
F(X) &:= \{b\in B\mid b\notin X+\langle B\downdownarrows b\rangle\}\,, && \forall X\in\operatorname{Sub}V\,;\\
f(X) &:= \langle F(X)\rangle\,, && \forall X\in\operatorname{Sub}V\,.
\end{aligned}$$

- Then $X\subseteq Y$ obviously implies that $F(Y)\subseteq F(X)$, thus $f(Y)\subseteq f(X)$.
- Verify that $X\cap f(X)=\{0\}$ (uses $\lhd$ linear ordering).
- Verify, by induction on $b\in B$, that $b\in X+f(X)$ (uses $\lhd$ well-ordering). Thus $V=X+f(X)$.

Proof of Banaschewski's Theorem

- Denote by $\lhd$ a strict well-ordering of a basis B of V. We set

$$\begin{aligned} \langle X\rangle &:= \text{subspace of } V \text{ generated by } X\,, && \forall X\in \operatorname{Sub} V\,;\\ B\downdownarrows b &:= \{x\in B \mid x\lhd b\}\,, && \forall b\in B\,;\\ F(X) &:= \{b\in B \mid b\notin X+\langle B\downdownarrows b\rangle\}\,, && \forall X\in \operatorname{Sub} V\,;\\ f(X) &:= \langle F(X)\rangle\,, && \forall X\in \operatorname{Sub} V\,. \end{aligned}$$

- Then $X\subseteq Y$ obviously implies that $F(Y)\subseteq F(X)$, thus $f(Y)\subseteq f(X)$.
- Verify that $X\cap f(X)=\{0\}$ (uses $\lhd$ linear ordering).
- Verify, by induction on $b\in B$, that $b\in X+f(X)$ (uses $\lhd$ well-ordering). Thus $V=X+f(X)$.
- Therefore, f is a Banaschewski function on $\operatorname{Sub} V$.

The ranges of those Banaschewski functions

- In the previous proof, $f(X) = \langle F(X) \rangle$, where $F(X) \subseteq B$.

The ranges of those Banaschewski functions

- In the previous proof, $f(X) = \langle F(X) \rangle$, where $F(X) \subseteq B$.
- Hence the range of f is $\{\langle X \rangle \mid X \subseteq B\}$.

The ranges of those Banaschewski functions

- In the previous proof, $f(X) = \langle F(X) \rangle$, where $F(X) \subseteq B$.
- Hence the range of f is $\{\langle X \rangle \mid X \subseteq B\}$.
- It is a Boolean algebra.

The ranges of those Banaschewski functions

- In the previous proof, $f(X) = \langle F(X) \rangle$, where $F(X) \subseteq B$.
- Hence the range of f is $\{\langle X \rangle \mid X \subseteq B\}$.
- It is a Boolean algebra.
- There are many such Boolean subalgebras of $\operatorname{Sub} V$, but they are all isomorphic (to the powerset of $\dim V$).

The ranges of those Banaschewski functions

- In the previous proof, $f(X) = \langle F(X) \rangle$, where $F(X) \subseteq B$.
- Hence the range of f is $\{\langle X \rangle \mid X \subseteq B\}$.
- It is a Boolean algebra.
- There are many such Boolean subalgebras of $\operatorname{Sub} V$, but they are all isomorphic (to the powerset of $\dim V$).
- How general is that phenomenon?

A lattice-theoretical viewpoint

- A lattice is a partially ordered set $(L, \leq)$ such that both $x \vee y := \sup\{x, y\}$ and $x \wedge y := \inf\{x, y\}$ exist for all $x, y \in L$.

A lattice-theoretical viewpoint

- A lattice is a partially ordered set $(L, \leq)$ such that both $x \vee y := \sup\{x, y\}$ and $x \wedge y := \inf\{x, y\}$ exist for all $x, y \in L$.
- We denote by 0 (resp., 1) the smallest (resp., largest) element if it exists. If both exist, we say that L is bounded.

A lattice-theoretical viewpoint

- A lattice is a partially ordered set $(L, \leq)$ such that both $x \vee y := \sup\{x, y\}$ and $x \wedge y := \inf\{x, y\}$ exist for all $x, y \in L$.
- We denote by 0 (resp., 1) the smallest (resp., largest) element if it exists. If both exist, we say that L is bounded.
- A complement of an element $a \in L$ is an element $b \in L$ such that $a \vee b = 1$ and $a \wedge b = 0$.

A lattice-theoretical viewpoint

- A lattice is a partially ordered set $(L, \leq)$ such that both $x \vee y := \sup\{x, y\}$ and $x \wedge y := \inf\{x, y\}$ exist for all $x, y \in L$.
- We denote by 0 (resp., 1) the smallest (resp., largest) element if it exists. If both exist, we say that L is bounded.
- A complement of an element $a \in L$ is an element $b \in L$ such that $a \vee b = 1$ and $a \wedge b = 0$.

Definition

A lattice-theoretical viewpoint

- A lattice is a partially ordered set $(L, \leq)$ such that both $x \vee y := \sup\{x, y\}$ and $x \wedge y := \inf\{x, y\}$ exist for all $x, y \in L$.
- We denote by 0 (resp., 1) the smallest (resp., largest) element if it exists. If both exist, we say that L is bounded.
- A complement of an element $a \in L$ is an element $b \in L$ such that $a \vee b = 1$ and $a \wedge b = 0$.

Definition

A Banaschewski function on a bounded lattice L is an antitone (=order-reversing) map $f : L \to L$ such that $f(x)$ is a complement of x, $\forall x \in L$.

A lattice-theoretical viewpoint

- A lattice is a partially ordered set $(L, \leq)$ such that both $x \vee y := \sup\{x, y\}$ and $x \wedge y := \inf\{x, y\}$ exist for all $x, y \in L$.
- We denote by 0 (resp., 1) the smallest (resp., largest) element if it exists. If both exist, we say that L is bounded.
- A complement of an element $a \in L$ is an element $b \in L$ such that $a \vee b = 1$ and $a \wedge b = 0$.

Definition

A Banaschewski function on a bounded lattice L is an antitone (=order-reversing) map $f \colon L \to L$ such that $f(x)$ is a complement of x, $\forall x \in L$.

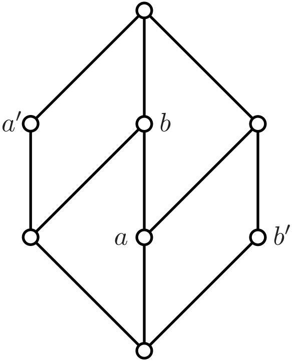
Hence $\operatorname{Sub} V$ has a Banaschewski function, for every vector space V.

A complemented lattice without a Banaschewski function

In the following lattice, every element has a complement (we say that L is complemented), but there is no Banaschewski function.

A complemented lattice without a Banaschewski function

In the following lattice, every element has a complement (we say that L is complemented), but there is no Banaschewski function.



Countable complemented modular lattices

- A lattice L is modular if $x \wedge (y \vee (x \wedge z)) = (x \wedge y) \vee (x \wedge z)$, $\forall x, y, z \in L$.

Countable complemented modular lattices

- A lattice L is modular if $x \wedge (y \vee (x \wedge z)) = (x \wedge y) \vee (x \wedge z)$, $\forall x, y, z \in L$.
- For example, $\operatorname{Sub} V$ is modular, for any vector space V.

Countable complemented modular lattices

- A lattice L is modular if $x \wedge (y \vee (x \wedge z)) = (x \wedge y) \vee (x \wedge z)$, $\forall x, y, z \in L$.
- For example, Sub V is modular, for any vector space V.
- More generally, $\mathbb{L}(R) := \{xR \mid x \in R\}$ is a complemented modular lattice, for every regular ring R.

Countable complemented modular lattices

- A lattice L is modular if $x \wedge (y \vee (x \wedge z)) = (x \wedge y) \vee (x \wedge z)$, $\forall x, y, z \in L$.
- For example, Sub V is modular, for any vector space V.
- More generally, $\mathbb{L}(R) := \{xR \mid x \in R\}$ is a complemented modular lattice, for every regular ring R.

Theorem (W. 2009)

Countable complemented modular lattices

- A lattice L is modular if $x \wedge (y \vee (x \wedge z)) = (x \wedge y) \vee (x \wedge z)$, $\forall x, y, z \in L$.
- For example, Sub V is modular, for any vector space V.
- More generally, $\mathbb{L}(R) := \{xR \mid x \in R\}$ is a complemented modular lattice, for every regular ring R.

Theorem (W. 2009)

Every countable complemented modular lattice has a Banaschewski function with Boolean range. This Boolean range is unique up to isomorphism.

Countable complemented modular lattices

- A lattice L is modular if $x \wedge (y \vee (x \wedge z)) = (x \wedge y) \vee (x \wedge z)$, $\forall x, y, z \in L$.
- For example, Sub V is modular, for any vector space V.
- More generally, $\mathbb{L}(R) := \{xR \mid x \in R\}$ is a complemented modular lattice, for every regular ring R.

Theorem (W. 2009)

Every countable complemented modular lattice has a Banaschewski function with Boolean range. This Boolean range is unique up to isomorphism.

Theorem (W. 2009)

Countable complemented modular lattices

- A lattice L is modular if $x \wedge (y \vee (x \wedge z)) = (x \wedge y) \vee (x \wedge z)$, $\forall x, y, z \in L$.
- For example, Sub V is modular, for any vector space V.
- More generally, $\mathbb{L}(R) := \{xR \mid x \in R\}$ is a complemented modular lattice, for every regular ring R.

Theorem (W. 2009)

Every countable complemented modular lattice has a Banaschewski function with Boolean range. This Boolean range is unique up to isomorphism.

Theorem (W. 2009)

There exists a unit-regular ring R, of index of nilpotence 3, of cardinality $\aleph_1$, such that $\mathbb{L}(R)$ has no Banaschewski function.

Banaschewski functions and countable regular rings

- The first result above is especially interesting when applied to $\mathbb{L}(R)$, for a countable regular ring R.

Banaschewski functions and countable regular rings

- The first result above is especially interesting when applied to $\mathbb{L}(R)$, for a countable regular ring R.
- It yields a Boolean sublattice $\mathbf{B}$ of $\mathbb{L}(R)$ and a Banaschewski function f with range $\mathbf{B}$.

Banaschewski functions and countable regular rings

- The first result above is especially interesting when applied to $\mathbb{L}(R)$, for a countable regular ring R.
- It yields a Boolean sublattice $\mathbf{B}$ of $\mathbb{L}(R)$ and a Banaschewski function f with range $\mathbf{B}$.
- For each $\mathbf{a} \in \mathbf{B}$, with complement $\mathbf{a}' \in \mathbf{B}$, $R = \mathbf{a} \oplus \mathbf{a}'$ as right R-modules.

Banaschewski functions and countable regular rings

- The first result above is especially interesting when applied to $\mathbb{L}(R)$, for a countable regular ring R.
- It yields a Boolean sublattice $\mathbf{B}$ of $\mathbb{L}(R)$ and a Banaschewski function f with range $\mathbf{B}$.
- For each $\mathbf{a} \in \mathbf{B}$, with complement $\mathbf{a}' \in \mathbf{B}$, $R = \mathbf{a} \oplus \mathbf{a}'$ as right R-modules.
- Thus there exists a unique pair $(a, a') \in \mathbf{a} \times \mathbf{a}'$ such that $1 = a + a'$. Note that $\mathbf{a} = aR$ and $\mathbf{a}' = a'R$.

Banaschewski functions and countable regular rings

- The first result above is especially interesting when applied to $\mathbb{L}(R)$, for a countable regular ring R.
- It yields a Boolean sublattice $\mathbf{B}$ of $\mathbb{L}(R)$ and a Banaschewski function f with range $\mathbf{B}$.
- For each $\mathbf{a} \in \mathbf{B}$, with complement $\mathbf{a}' \in \mathbf{B}$, $R = \mathbf{a} \oplus \mathbf{a}'$ as right R-modules.
- Thus there exists a unique pair $(a, a') \in \mathbf{a} \times \mathbf{a}'$ such that $1 = a + a'$. Note that $\mathbf{a} = aR$ and $\mathbf{a}' = a'R$.
- Set $B := \{a \mid \mathbf{a} \in \mathbf{B}\}$. Then $\mathbf{B} = \{aR \mid a \in B\}$.

Banaschewski functions and countable regular rings

- The first result above is especially interesting when applied to $\mathbb{L}(R)$, for a countable regular ring R.
- It yields a Boolean sublattice $\mathbf{B}$ of $\mathbb{L}(R)$ and a Banaschewski function f with range $\mathbf{B}$.
- For each $\mathbf{a} \in \mathbf{B}$, with complement $\mathbf{a}' \in \mathbf{B}$, $R = \mathbf{a} \oplus \mathbf{a}'$ as right R-modules.
- Thus there exists a unique pair $(a, a') \in \mathbf{a} \times \mathbf{a}'$ such that $1 = a + a'$. Note that $\mathbf{a} = aR$ and $\mathbf{a}' = a'R$.
- Set $B := \{a \mid \mathbf{a} \in \mathbf{B}\}$. Then $\mathbf{B} = \{aR \mid a \in B\}$.
- Furthermore, B is a Boolean algebra of idempotents of R: this means that B consists of pairwise commuting idempotents, $0 \in B$, and B is closed under $a \mapsto 1 - a$ and $(a, b) \mapsto ab$.

Banaschewski functions and countable regular rings (cont'd)

- Actually, B is a maximal commutative set of idempotents (MCSI) in R.

Banaschewski functions and countable regular rings (cont'd)

- Actually, B is a maximal commutative set of idempotents (MCSI) in R.
- This B (obtained *via* a Banaschewski function) is unique up to isomorphism (more detail later).

Banaschewski functions and countable regular rings (cont'd)

- Actually, B is a maximal commutative set of idempotents (MCSI) in R.
- This B (obtained *via* a Banaschewski function) is unique up to isomorphism (more detail later).
- Is there any associated maximal abelian regular subring (MARS) of R?

Banaschewski functions and countable regular rings (cont'd)

- Actually, B is a maximal commutative set of idempotents (MCSI) in R.
- This B (obtained *via* a Banaschewski function) is unique up to isomorphism (more detail later).
- Is there any associated maximal abelian regular subring (MARS) of R?
- A key property is that (for that particular B) $\forall x \in R$, $\exists a \in B$ such that $R = xR \oplus aR$.

Banaschewski functions and countable regular rings (cont'd)

- Actually, B is a maximal commutative set of idempotents (MCSI) in R.
- This B (obtained *via* a Banaschewski function) is unique up to isomorphism (more detail later).
- Is there any associated maximal abelian regular subring (MARS) of R?
- A key property is that (for that particular B) $\forall x \in R$, $\exists a \in B$ such that $R = xR \oplus aR$.

Proposition (W. 2010)

Banaschewski functions and countable regular rings (cont'd)

- Actually, B is a maximal commutative set of idempotents (MCSI) in R.
- This B (obtained *via* a Banaschewski function) is unique up to isomorphism (more detail later).
- Is there any associated maximal abelian regular subring (MARS) of R?
- A key property is that (for that particular B) $\forall x \in R$, $\exists a \in B$ such that $R = xR \oplus aR$.

Proposition (W. 2010)

Let B be a Boolean algebra of idempotents in a regular ring R, such that $\forall x \in R$, $\exists a \in B$ such that $R = xR \oplus aR$.

Banaschewski functions and countable regular rings (cont'd)

- Actually, B is a maximal commutative set of idempotents (MCSI) in R.
- This B (obtained *via* a Banaschewski function) is unique up to isomorphism (more detail later).
- Is there any associated maximal abelian regular subring (MARS) of R?
- A key property is that (for that particular B) $\forall x \in R$, $\exists a \in B$ such that $R = xR \oplus aR$.

Proposition (W. 2010)

Let B be a Boolean algebra of idempotents in a regular ring R, such that $\forall x \in R$, $\exists a \in B$ such that $R = xR \oplus aR$. Then the commutant of B is a MARS of R, with set of idempotents B.

Banaschewski functions and countable regular rings (cont'd)

- Actually, B is a maximal commutative set of idempotents (MCSI) in R.
- This B (obtained *via* a Banaschewski function) is unique up to isomorphism (more detail later).
- Is there any associated maximal abelian regular subring (MARS) of R?
- A key property is that (for that particular B) $\forall x \in R$, $\exists a \in B$ such that $R = xR \oplus aR$.

Proposition (W. 2010)

Let B be a Boolean algebra of idempotents in a regular ring R, such that $\forall x \in R$, $\exists a \in B$ such that $R = xR \oplus aR$. Then the commutant of B is a MARS of R, with set of idempotents B.

Interesting for starting a Boolean-valued analysis of the ring R.

The canonical V-measure on B

- Let R be a countable regular ring, let $\mathbf{B} \subseteq \mathbb{L}(R)$ be the range of a Banaschewski function on $\mathbb{L}(R)$, and let B be the associated MCSI (so $\mathbf{B} = \{aR \mid a \in B\}$).

The canonical V-measure on B

- Let R be a countable regular ring, let $\mathbf{B} \subseteq \mathbb{L}(R)$ be the range of a Banaschewski function on $\mathbb{L}(R)$, and let B be the associated MCSI (so $\mathbf{B} = \{aR \mid a \in B\}$).
- Consider $\mu\colon B \to \mathbb{V}(R)$, $a \mapsto [aR]$.

The canonical V-measure on B

- Let R be a countable regular ring, let $\mathbf{B} \subseteq \mathbb{L}(R)$ be the range of a Banaschewski function on $\mathbb{L}(R)$, and let B be the associated MCSI (so $\mathbf{B} = \{aR \mid a \in B\}$).
- Consider $\mu\colon B \to \mathbb{V}(R)$, $a \mapsto [aR]$.
- Then $\mu(x) = 0 \Leftrightarrow x = 0$ and $\mu(a+b) = \mu(a) + \mu(b)$ for any disjoint $a, b \in B$.

The canonical V-measure on B

- Let R be a countable regular ring, let $\mathbf{B} \subseteq \mathbb{L}(R)$ be the range of a Banaschewski function on $\mathbb{L}(R)$, and let B be the associated MCSI (so $\mathbf{B} = \{aR \mid a \in B\}$).
- Consider $\mu\colon B \to \mathbb{V}(R)$, $a \mapsto [aR]$.
- Then $\mu(x) = 0 \Leftrightarrow x = 0$ and $\mu(a + b) = \mu(a) + \mu(b)$ for any disjoint $a, b \in B$. Furthermore, $\mu(1) = [R]$.

The canonical V-measure on B

- Let R be a countable regular ring, let $\mathbf{B} \subseteq \mathbb{L}(R)$ be the range of a Banaschewski function on $\mathbb{L}(R)$, and let B be the associated MCSI (so $\mathbf{B} = \{aR \mid a \in B\}$).
- Consider $\mu\colon B \to \mathbb{V}(R)$, $a \mapsto [aR]$.
- Then $\mu(x) = 0 \Leftrightarrow x = 0$ and $\mu(a+b) = \mu(a) + \mu(b)$ for any disjoint $a, b \in B$. Furthermore, $\mu(1) = [R]$.
- So μ is a finitely additive probability measure on the Boolean algebra B, with values in the monoid $\mathbb{V}(R)$.

The canonical V-measure on B

- Let R be a countable regular ring, let $\mathbf{B} \subseteq \mathbb{L}(R)$ be the range of a Banaschewski function on $\mathbb{L}(R)$, and let B be the associated MCSI (so $\mathbf{B} = \{aR \mid a \in B\}$).
- Consider $\mu\colon B \to \mathbb{V}(R)$, $a \mapsto [aR]$.
- Then $\mu(x) = 0 \Leftrightarrow x = 0$ and $\mu(a+b) = \mu(a) + \mu(b)$ for any disjoint $a, b \in B$. Furthermore, $\mu(1) = [R]$.
- So μ is a finitely additive probability measure on the Boolean algebra B, with values in the monoid $\mathbb{V}(R)$.
- Additional property of μ: it satisfies the V-condition, that is,

The canonical V-measure on B

- Let R be a countable regular ring, let $\mathbf{B} \subseteq \mathbb{L}(R)$ be the range of a Banaschewski function on $\mathbb{L}(R)$, and let B be the associated MCSI (so $\mathbf{B} = \{aR \mid a \in B\}$).
- Consider $\mu\colon B \to \mathbb{V}(R)$, $a \mapsto [aR]$.
- Then $\mu(x) = 0 \Leftrightarrow x = 0$ and $\mu(a+b) = \mu(a) + \mu(b)$ for any disjoint $a, b \in B$. Furthermore, $\mu(1) = [R]$.
- So μ is a finitely additive probability measure on the Boolean algebra B, with values in the monoid $\mathbb{V}(R)$.
- Additional property of μ: it satisfies the V-condition, that is,

$$\mu(c) = \alpha+\beta \Rightarrow (\exists a, b)\bigl(c = a \oplus b \mathbin{\&} \mu(a) = \alpha \mathbin{\&} \mu(b) = \beta\bigr)\,.$$

The canonical V-measure on B

- Let R be a countable regular ring, let $\mathbf{B} \subseteq \mathbb{L}(R)$ be the range of a Banaschewski function on $\mathbb{L}(R)$, and let B be the associated MCSI (so $\mathbf{B} = \{aR \mid a \in B\}$).
- Consider $\mu\colon B \to \mathbb{V}(R)$, $a \mapsto [aR]$.
- Then $\mu(x) = 0 \Leftrightarrow x = 0$ and $\mu(a + b) = \mu(a) + \mu(b)$ for any disjoint $a, b \in B$. Furthermore, $\mu(1) = [R]$.
- So μ is a finitely additive probability measure on the Boolean algebra B, with values in the monoid $\mathbb{V}(R)$.
- Additional property of μ: it satisfies the V-condition, that is,

$$\mu(c) = \alpha+\beta \Rightarrow (\exists a, b)\big(c = a\oplus b \;\&\; \mu(a) = \alpha \;\&\; \mu(b) = \beta\big)\,.$$

- We say that μ is a V-measure on B.

From the V-measure to the uniqueness of the Boolean range

A V-relation between Boolean algebras A and B is a binary relation $\rho \subseteq A \times B$ such that $1_A \, \rho \, 1_B$, $a \, \rho \, 0_B \Leftrightarrow a = 0_A$, $a \, \rho \, b_0 \oplus b_1 \Rightarrow \exists a_0, a_1$ such that $a = a_0 \oplus a_1$ and $a_i \, \rho \, b_i \; \forall i < 2$, and similarly with A and B interchanged.

From the V-measure to the uniqueness of the Boolean range

A V-relation between Boolean algebras A and B is a binary relation $\rho \subseteq A \times B$ such that $1_A \, \rho \, 1_B$, $a \, \rho \, 0_B \Leftrightarrow a = 0_A$, $a \, \rho \, b_0 \oplus b_1 \Rightarrow \exists a_0, a_1$ such that $a = a_0 \oplus a_1$ and $a_i \, \rho \, b_i \; \forall i < 2$, and similarly with A and B interchanged.

Theorem (Vaught 1954)

From the V-measure to the uniqueness of the Boolean range

A V-relation between Boolean algebras A and B is a binary relation $\rho \subseteq A \times B$ such that $1_A \, \rho \, 1_B$, $a \, \rho \, 0_B \Leftrightarrow a = 0_A$, $a \, \rho \, b_0 \oplus b_1 \Rightarrow \exists a_0, a_1$ such that $a = a_0 \oplus a_1$ and $a_i \, \rho \, b_i \; \forall i < 2$, and similarly with A and B interchanged.

Theorem (Vaught 1954)

Every V-relation between countable Boolean algebras A and B contains the graph of some isomorphism $A \to B$.

From the V-measure to the uniqueness of the Boolean range

A V-relation between Boolean algebras A and B is a binary relation $\rho \subseteq A \times B$ such that $1_A \, \rho \, 1_B$, $a \, \rho \, 0_B \Leftrightarrow a = 0_A$, $a \, \rho \, b_0 \oplus b_1 \Rightarrow \exists a_0, a_1$ such that $a = a_0 \oplus a_1$ and $a_i \, \rho \, b_i \; \forall i < 2$, and similarly with A and B interchanged.

Theorem (Vaught 1954)

Every V-relation between countable Boolean algebras A and B contains the graph of some isomorphism $A \to B$.

Now for Boolean algebras A and B, an element e in a conical refinement monoid M, and V-measures $\mu \colon A \to M$ and $\nu \colon B \to M$ with $\mu(1) = \nu(1) = e$, the binary relation

$$\{(a, b) \in A \times B \mid \mu(a) = \nu(b)\}$$

is a V-relation.

Dobbertin's Theorem

From this we obtain the uniqueness statement in the following representation result for any conical refinement monoid with order-unit.

Dobbertin's Theorem

From this we obtain the uniqueness statement in the following representation result for any conical refinement monoid with order-unit.

Theorem (Dobbertin 1983)

Dobbertin's Theorem

From this we obtain the uniqueness statement in the following representation result for any conical refinement monoid with order-unit.

Theorem (Dobbertin 1983)

For every element e in a countable conical refinement monoid M, there are a countable Boolean algebra B and a V-measure $\mu\colon B \to M$ such that $\mu(1) = e$. This measure is unique up to isomorphism

Dobbertin's Theorem

From this we obtain the uniqueness statement in the following representation result for any conical refinement monoid with order-unit.

Theorem (Dobbertin 1983)

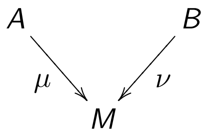
For every element e in a countable conical refinement monoid M, there are a countable Boolean algebra B and a V-measure $\mu\colon B \to M$ such that $\mu(1) = e$. This measure is unique up to isomorphism (but usually not up to unique isomorphism).

Dobbertin's Theorem

From this we obtain the uniqueness statement in the following representation result for any conical refinement monoid with order-unit.

Theorem (Dobbertin 1983)

For every element e in a countable conical refinement monoid M, there are a countable Boolean algebra B and a V-measure $\mu\colon B \to M$ such that $\mu(1) = e$. This measure is unique up to isomorphism (but usually not up to unique isomorphism).



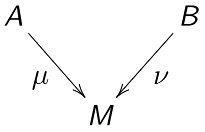
Two V-measures. . .

Dobbertin's Theorem

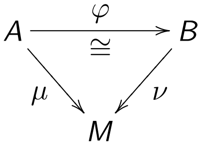
From this we obtain the uniqueness statement in the following representation result for any conical refinement monoid with order-unit.

Theorem (Dobbertin 1983)

For every element e in a countable conical refinement monoid M, there are a countable Boolean algebra B and a V-measure $\mu\colon B \to M$ such that $\mu(1) = e$. This measure is unique up to isomorphism (but usually not up to unique isomorphism).



Two V-measures...



...are isomorphic

Banaschewski functions again

- The uniqueness of the Boolean range of a Banaschewski function on $\mathbb{L}(R)$, R countable regular, follows immediately.

Banaschewski functions again

- The uniqueness of the Boolean range of a Banaschewski function on $\mathbb{L}(R)$, R countable regular, follows immediately.
- Extends to countable complemented modular lattices: the analogue of $\mathbb{V}(R)$ is the dimension monoid $\operatorname{Dim} L$ (for a lattice L).

Banaschewski functions again

- The uniqueness of the Boolean range of a Banaschewski function on $\mathbb{L}(R)$, R countable regular, follows immediately.
- Extends to countable complemented modular lattices: the analogue of $\mathbb{V}(R)$ is the dimension monoid $\operatorname{Dim} L$ (for a lattice L). For a regular ring $R \cong \mathrm{M}_2(R')$, $\mathbb{V}(R) \cong \operatorname{Dim} \mathbb{L}(R)$.
- For refinement monoids and Boolean algebras with $\leq \aleph_1$ elements, the existence part of Dobbertin's Theorem remains, but the uniqueness part is lost (Dobbertin 1983).

Banaschewski functions again

- The uniqueness of the Boolean range of a Banaschewski function on $\mathbb{L}(R)$, R countable regular, follows immediately.
- Extends to countable complemented modular lattices: the analogue of $\mathbb{V}(R)$ is the dimension monoid $\operatorname{Dim} L$ (for a lattice L). For a regular ring $R \cong \mathrm{M}_2(R')$, $\mathbb{V}(R) \cong \operatorname{Dim} \mathbb{L}(R)$.
- For refinement monoids and Boolean algebras with $\leq \aleph_1$ elements, the existence part of Dobbertin's Theorem remains, but the uniqueness part is lost (Dobbertin 1983).
- For refinement monoids with $\geq \aleph_2$ elements, both existence and uniqueness in Dobbertin's Theorem are lost (W. 1998).

A strategy of approach of the Realization Problem. . .

- Start with a countable conical refinement monoid M with order-unit e.

A strategy of approach of the Realization Problem. . .

- Start with a countable conical refinement monoid M with order-unit e.
- Let $\mu\colon B \to M$ be the unique V-measure, for a countable Boolean algebra B, with $\mu(1) = e$.

A strategy of approach of the Realization Problem. . .

- Start with a countable conical refinement monoid M with order-unit e.
- Let $\mu\colon B \to M$ be the unique V-measure, for a countable Boolean algebra B, with $\mu(1) = e$.
- Develop a Boolean-valued analysis of a countable regular ring R with a MCSI $B \subseteq R$ associated with a Banaschewski function with Boolean range on $\mathbb{L}(R)$.

A strategy of approach of the Realization Problem. . .

- Start with a countable conical refinement monoid M with order-unit e.
- Let $\mu\colon B \to M$ be the unique V-measure, for a countable Boolean algebra B, with $\mu(1) = e$.
- Develop a Boolean-valued analysis of a countable regular ring R with a MCSI $B \subseteq R$ associated with a Banaschewski function with Boolean range on $\mathbb{L}(R)$.
- Try to re-create the structure thus guessed, now starting again from $\mu\colon B \to M$. . .

. . . nobody knows. . .