Spectral spaces of countable Abelian ℓ-groups

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Equivalent formulation: describe the spectra of MV-algebras.

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- A topological space X is generalized spectral if it is sober (i.e., every join-irreducible closed set is the closure of a unique singleton) and the set $\overset{\circ}{\mathcal{K}}(X)$ of all compact open subsets of X is a basis of the topology of X, closed under $(U, V) \mapsto U \cap V$.

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- If, in addition, X is compact, then we say that X is spectral.

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By definition, a map $\varphi \colon X \to Y$ is spectral if $\forall V \in \overset{\circ}{\mathcal{K}}(Y)$, $\varphi^{-1}[V] \in \overset{\circ}{\mathcal{K}}(X)$.

The lattice $\mathrm{Id_c}\, G$

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- Recasts the above problem as: Describe ℓ-representable lattices.

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Theorem (Monteiro 1956)

A generalized spectral space X is completely normal iff the distributive lattice $\overset{\circ}{\mathcal{K}}(X)$ is completely normal.

Complete normality of $\mathrm{Id_c}\, G$

Proposition (folklore)

For every Abelian ℓ-group G, $\mathrm{Id_c}\, G$ is a completely normal distributive lattice (equivalently, $\mathrm{Spec}_\ell\, G$ is a completely normal generalized spectral space).

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Proof.

Let $\boldsymbol{a}, \boldsymbol{b} \in \mathrm{Id_c}\, G$. There are $a, b \in G^+$ such that $\boldsymbol{a} = \langle a \rangle$ and $\boldsymbol{b} = \langle b \rangle$. Set $\boldsymbol{x} \underset{\mathrm{def}}{=} \langle a - a \wedge b \rangle$ and $\boldsymbol{y} \underset{\mathrm{def}}{=} \langle b - a \wedge b \rangle$. Then $(\boldsymbol{x}, \boldsymbol{y})$ is a splitting of $(\boldsymbol{a}, \boldsymbol{b})$. □

Countably based differences

Definition

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Then $\{\boldsymbol{c}_n \mid n < \omega\}$ is coinitial in $\boldsymbol{a} \ominus \boldsymbol{b}$. □

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- By using a different construction, $2^{\aleph_1}$ can be improved to $\aleph_1$ (W 2017).

No $\mathscr{L}_{\infty,\omega}$ characterization of ℓ-representability

- Set $\boldsymbol{B}_I \underset{\text{def}}{=} \{X \subseteq I \mid X \text{ or } I \setminus X \text{ is finite}\}$ and

$$\boldsymbol{D}_I \underset{\text{def}}{=} \{(X,k) \in \boldsymbol{B}_I \times \{0,1,2\} \mid (k = 0 \Rightarrow X \text{ finite}) \text{ and } (k \neq 0 \Rightarrow I \setminus X \text{ finite})\}\,,$$

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- $\boldsymbol{D}_\omega \hookrightarrow \boldsymbol{D}_{\omega_1}$, *via*

$$(X,k) \mapsto \begin{cases} (X,k)\,, & \text{if } k=0\,,\\ (X \cup (\omega_1 \setminus \omega), k)\,, & \text{if } k \neq 0\,. \end{cases}$$

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$\boldsymbol{D}_\omega$ is an $\mathscr{L}_{\infty,\omega}$-elementary sublattice of $\boldsymbol{D}_{\omega_1}$ (use back-and-forth), with $\boldsymbol{D}_\omega$ countable (and ℓ-representable) and $\boldsymbol{D}_{\omega_1}$ non-ℓ-representable (no countably based differences).

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The proof extends (non-trivially) the finite case. In that case, D is the lattice of all lower subsets of a finite root system P. So $D \cong \mathrm{Id_c}\, \mathbb{Q}\langle P \rangle$, where $\mathbb{Q}\langle P \rangle$ is the lexicographical power (Hahn power) of $\mathbb{Q}$ by P.

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Definition

For distributive lattices D and E with zero, a 0-lattice homomorphism $f\colon D \to E$ is closed if for all $a, b \in D$ and all $c \in E$, $f(a) \leq f(b) \vee c \Rightarrow \exists x \in D,\ a \leq b \vee x$ and $f(x) \leq c$.

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This means that $f(a) \leq nf(b) + nc$, for some $n < \omega$.
Hence $f(x) \leq nc$, where $x \underset{\mathrm{def}}{=} a - (a \wedge nb)$.
Therefore, $\langle a\rangle \subseteq \langle b\rangle \vee \langle x\rangle$, with $(\mathrm{Id_c}\, f)(\langle x\rangle) \subseteq \langle c\rangle$. □

Closed lattice homomorphisms (cont'd)

Proposition

Let G be an Abelian ℓ-group, let $\boldsymbol{D}$ be a distributive lattice with zero. Then every surjective closed lattice homomorphism $\boldsymbol{f}\colon \mathrm{Id_c}\, G \twoheadrightarrow \boldsymbol{D}$ induces an isomorphism $\mathrm{Id_c}\,(G/I) \to \boldsymbol{D}$, for the ℓ-ideal $I \underset{\mathrm{def}}{=} \{x \in G \mid \boldsymbol{f}(\langle x \rangle) = 0\}$.

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A 2004 example by Di Nola and Grigolia shows that the E_n cannot always be taken completely normal.

Defining $\mathrm{Op}(\mathcal{H})$

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Let $\mathcal{H}$ be a set of closed hyperplanes in a topological vector space $\mathbb{E}$ over $\mathbb{R}$.

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$$\text{generated by all } H^+ \text{ and } H^-\,, \text{ where } H \in \mathcal{H}\,;$$

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Lemma

For every $X \in \mathrm{Bool}(\mathcal{H})$, $\mathrm{int}(X)$ belongs to $\mathrm{Op}(\mathcal{H})$, and it is a finite union of sets of the form $\bigcap_{i=1}^{n} H_i^{\pm}$, where all $H_i \in \mathcal{H}$ (basic open sets).

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For $U \in \mathrm{Op}(\mathcal{H})$, we set

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Thus, ∇U is a closed subspace of $\mathbb{E}$, with finite codimension.

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A convex member P of $\operatorname{Op}(\mathcal{H})$ is join-irreducible iff $P \cap \nabla P \neq \varnothing$, in which case $P_* = P \setminus \nabla P$ and $P^\dagger = \complement(\operatorname{cl}(P) \cap \nabla P) = \complement \operatorname{cl}(P \cap \nabla P)$ (the largest $X \in \operatorname{Op}(\mathcal{H})$ such that $P \not\subseteq X$).

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- Recall that in any finite distributive lattice D, $p \mapsto p^{\dagger}$ is an order-isomorphism between $\mathrm{Ji}\, D \underset{\mathrm{def}}{=} \{\text{join-irreducibles of } D\}$ and $\mathrm{Mi}\, D \underset{\mathrm{def}}{=} \{\text{meet-irreducibles of } D\}$ (with induced $\leq$ from D).

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- Important observation about $\mathrm{Op}(\mathcal{H})$: $P \setminus P_* = P \cap \nabla P$ is convex $\forall P \in \mathrm{Ji}\,\mathrm{Op}(\mathcal{H})$.

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Sketch of proof.

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(For that part of the proof, we may assume that X is join-irreducible.) □

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From $P \subsetneqq Q$ it follows that $P \subseteq Q_* = Q \setminus \nabla Q$, thus $P \cap \nabla Q = \varnothing$.
Since $P \cap \nabla P \neq \varnothing$, we get $\nabla P \neq \nabla Q$. □

Consonance

Definition

Let D be a distributive lattice with zero. Elements $a, b \in D$ are consonant, in notation $a \sim b$, if $\exists x, y \in D$ such that $a \leq b \vee x$, $b \leq a \vee y$, and $x \wedge y = 0$ (again: we say that (x, y) is a splitting of (a, b)).

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In particular, D is completely normal iff any two elements of D are consonant (i.e., D is a consonant subset of itself).

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1. $a \leq b \Rightarrow a \sim b$;
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Hence, $a \vee b \sim c$.

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Then $a \vee b \leq c \vee (x \vee y)$, $c \leq (a \vee b) \vee (x' \wedge y')$, and $(x \vee y) \wedge (x' \wedge y') = 0$.
Hence, $a \vee b \sim c$.
The proof that $a \wedge b \sim c$ is similar. □

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Lemma

Let L be finite. Then $a \smallsetminus b = \bigvee (p \smallsetminus p_* \mid p \in \mathrm{Ji}\, L\,,\ p \leq a\,,\ p \nleq b)$, $\forall a, b \in L$.

Pseudo-differences again

Lemma

Let D be a finite distributive lattice. Then the pseudo-difference, $(x, y) \mapsto x \smallsetminus_D y \underset{\text{def}}{=}$ least $z \in D$ such that $x \leq y \vee z$, is a D-valued difference operation on D, normal on every consonant sublattice of D.

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Now we state two lemmas that will be crucial for further computations.

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(1) is straightforward. Let us see (2).

$$\begin{aligned} a_1 \smallsetminus_D b &\leq \left(a_1 \smallsetminus_D (a_1 \wedge a_2)\right) \vee \left((a_1 \wedge a_2) \smallsetminus_D b\right) \\ &= (a_1 \smallsetminus_D a_2) \vee \left((a_1 \wedge a_2) \smallsetminus_D b\right). \end{aligned}$$

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Likewise, $a_2 \smallsetminus_D b \leq (a_2 \smallsetminus_D a_1) \vee \big((a_1 \wedge a_2) \smallsetminus_D b\big)$.

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$$a_1 \smallsetminus_D b \leq (a_1 \smallsetminus_D (a_1 \wedge a_2)) \vee ((a_1 \wedge a_2) \smallsetminus_D b)$$
$$= (a_1 \smallsetminus_D a_2) \vee ((a_1 \wedge a_2) \smallsetminus_D b)\,.$$

Likewise, $a_2 \smallsetminus_D b \leq (a_2 \smallsetminus_D a_1) \vee ((a_1 \wedge a_2) \smallsetminus_D b)$.
The relation $a_1 \sim a_2$ can be rewritten $(a_1 \smallsetminus_D a_2) \wedge (a_2 \smallsetminus_D a_1) = 0$.

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The relation $a_1 \sim a_2$ can be rewritten $(a_1 \smallsetminus_D a_2) \wedge (a_2 \smallsetminus_D a_1) = 0$.
Thus (distributivity) $(a_1 \smallsetminus_D b) \wedge (a_2 \smallsetminus_D b) \leq (a_1 \wedge a_2) \smallsetminus_D b$. □

Second crucial lemma

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If $a_1 \sim a_2$ and $a_1 \wedge a_2 \leq b_1 \wedge b_2$, then $(a_1 \searrow_D b_1) \wedge (a_2 \searrow_D b_2) = 0$.

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$$\begin{aligned}(a_1 \smallsetminus_D b_1) \wedge (a_2 \smallsetminus_D b_2) &\leq (a_1 \smallsetminus_D b) \wedge (a_2 \smallsetminus_D b)\\ &= (a_1 \wedge a_2) \smallsetminus_D b \qquad (\text{because } a_1 \sim a_2)\\ &= 0 \qquad (\text{because } a_1 \wedge a_2 \leq b).\end{aligned}$$

□

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Problem: we are given finite distributive lattices E and L, a $0,1$-sublattice D of E, and a 0-lattice homomorphism $f : D \to L$.

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Suppose that there are $a, b \in E$ such that the following statements hold:

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1. (The range of) f is consonant in L;
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4. $a \wedge b = 0$;
5. $\forall p \in \mathrm{Ji}\, D,\ p \leq p_* \vee a \vee b \Rightarrow (p \leq p_* \vee a \text{ or } p \leq p_* \vee b)$;
6. $\forall p, q \in \mathrm{Ji}\, D,\ (p \leq p_* \vee a \text{ and } q \leq q_* \vee b) \Rightarrow (p \text{ and } q \text{ are incomparable})$.

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Then such an extension g exists, with $g(a) = f_*(a)$ and $g(b) = f_*(b)$, where $f_*(t) = \bigvee \left(f(p) \smallsetminus_L f(p_*) \mid p \in \mathrm{Ji}\, D\,,\ p \leq p_* \vee t\right), \forall t \in E$.

Outline of proof (Extension Lemma)

- We want to define $g\bigl((x \wedge a) \vee (y \wedge b) \vee z\bigr) \underset{\mathrm{def}}{=} \bigl(f(x) \wedge f_*(a)\bigr) \vee \bigl(f(y) \wedge f_*(b)\bigr) \vee f(z)$ $\forall x, y, z \in D$. We must verify certain compatibility relations.

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- By Assumption (5), either $p \leq p_* \vee a$ or $p \leq p_* \vee b$.
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- Assumption (6) used for $f_*(a) \wedge f_*(b) = 0$.

The Extension Lemma for Op($\mathcal{H}$)

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Let $\mathcal{H}$ be a finite set of closed hyperplanes in a topological vector space $\mathbb{E}$, let H be a closed hyperplane of $\mathbb{E}$, and let L be a finite distributive lattice.

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Outline of proof. Verify one by one the conditions of the Extension Lemma for lattices, with $D := \mathrm{Op}(\mathcal{H})$, $E := \mathrm{Op}(\mathcal{H} \cup \{H\})$, $a := H^+$, and $b := H^-$.

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Outline of proof. Verify one by one the conditions of the Extension Lemma for lattices, with $D := \mathrm{Op}(\mathcal{H})$, $E := \mathrm{Op}(\mathcal{H} \cup \{H\})$, $a := H^+$, and $b := H^-$.

- Every basic open set in $\mathrm{Op}(\mathcal{H} \cup \{H\})$ has the form U or $U \cap H^\pm$, where U is basic open in Op($\mathcal{H}$); whence $E = D[a, b]$.

Extension Lemma for Op($\mathcal{H}$) (cont'd)

- Both $D = \mathrm{Op}(\mathcal{H})$ and $E = \mathrm{Op}(\mathcal{H} \cup \{H\})$ are Heyting subalgebras of the lattice of all open subsets of $\mathbb{E}$; whence D is a Heyting subalgebra of E.

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- Since $P \cap \nabla P$ is convex, either $P \cap \nabla P \subseteq H^+$ or $P \cap \nabla P \subseteq H^-$, that is, either $P \subseteq P_* \cup H^+$ or $P \subseteq P_* \cup H^-$.

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Extension Lemma for Op($\mathcal{H}$) (cont'd)

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- Hence $Q \cap \nabla Q \subseteq H^- \cap \overline{H}^+ = \varnothing$, a contradiction.

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- Given a countable, completely normal distributive lattice D with zero, construct inductively a closed, surjective lattice homomorphism $f = \bigcup_{n<\omega} f_n \colon \mathrm{Id_c}\, \mathrm{F}_\ell(\omega) \twoheadrightarrow D$, where (using Baker-Beynon duality) all $E_n = \mathrm{Op}^-(\mathcal{H}_n) \underset{\mathrm{def}}{=} \mathrm{Op}(\mathcal{H}_n) \setminus \{\mathbb{R}^{(\omega)}\}$ and $f_n \colon E_n \to D$.

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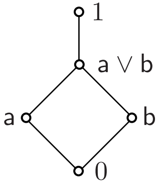
- (Ensuring f surjective) If H is "independent" from $\mathcal{H}$, then $\mathrm{Op}(\mathcal{H} \cup \{H\}) \cong \mathrm{Op}(\mathcal{H}) * \mathrm{J}_2$ (free distributive product), where J_2 is



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- "Correct any instance of $f(A^+) \leq f(B^+) \vee \gamma$".

Forcing closedness of a consonant homomorphism

Let $\mathbb{E} := \mathbb{R}^{(\omega)}$, with canonical inner product $(x|y) \underset{\mathrm{def}}{=} \sum_{n<\omega} x_n y_n$ and weak topology (making all $(x|\_)$ continuous).

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- Existence of m ensured by Farkas' Lemma (Hahn-Banach Theorem).

Convex ℓ-subgroups of ℓ-groups

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Uncountable analogue of corollary above: fails (Kenoyer 1984, McCleary 1986).

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- It is not known whether every second countable, completely normal spectral space is homeomorphic to the real spectrum of some commutative unital ring.