The equational theory of permutohedra

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Outline

1. Elementary theory of permutohedra
2. An identity satisfied by all the permutohedra
3. Decidability of the weak Bruhat ordering on permutations via MSOL and S1S
4. No identities for generalized permutohedra

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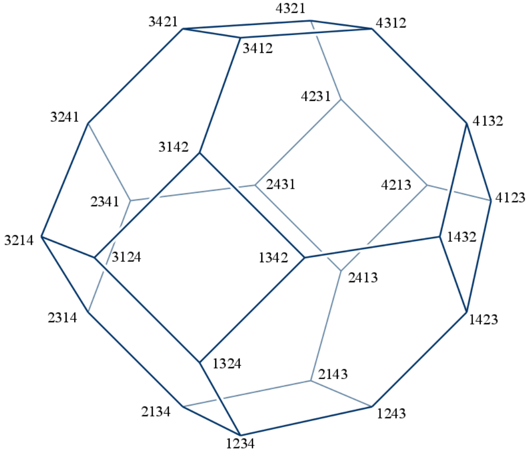
1. Elementary theory of permutohedra
   - Permutohedra
   - Cambrian lattices
   - Geyer's Conj
   - $\not\hookrightarrow$ A(N)
   - $\not\hookrightarrow$ P(N)
   - $\in$ HS(A$_U$(N))
2. An identity satisfied by all the permutohedra
3. Decidability of the weak Bruhat ordering on permutations via MSOL and S1S
4. No identities for generalized permutohedra

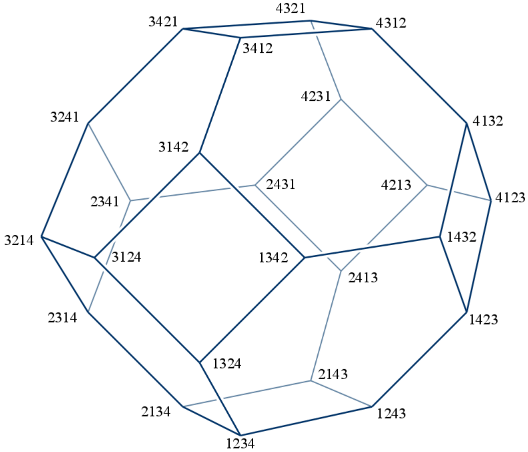
What is a permutohedron (I)?

P(N) := Cayley graph of the symmetric group group $\mathfrak{S}_N$



What is a permutohedron (I)?

P(N) := convex polytope of the symmetric group group $\mathfrak{S}_N$

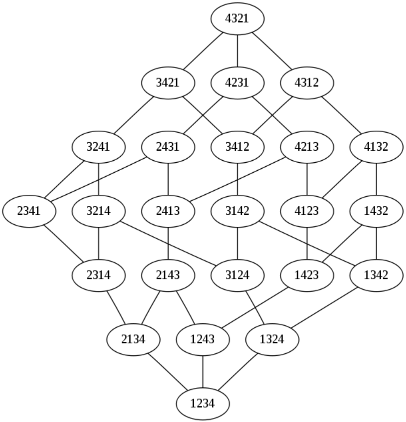


The weak Bruhat ordering

Obtained from the Cayley graph P(N) by:

1. directing edges along increasing length of a permutation;
2. taking the reflexive-transitive closure of this DAG.

P(4)



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$$[N] \underset{\text{def.}}{=} \{1, 2, \dots, N\}\,,$$

$$\mathfrak{I}_N \underset{\text{def.}}{=} \{(i,j) \in [N] \times [N] \mid i < j\}\,,$$

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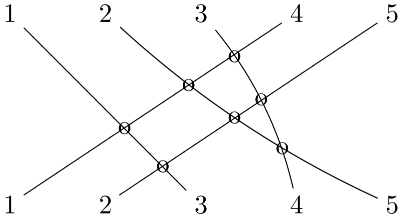
- Alternative definition of the permutohedron:

$$\mathrm{P}(N) := \{\mathrm{Inv}(\sigma) \mid \sigma \in \mathfrak{S}_N\},\ \text{ordered by } \subseteq.$$

What are the Inv(σ) (I)?

The a string diagram of the permutation 35412:



$$\mathsf{Inv}(\sigma) = \{(i,j) \mid (i,j) \text{ is a crossing on the string diagram of } \sigma\}$$

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- Conversely, every subset $\mathbf{x} \subseteq \mathcal{I}_N$, such that both $\mathbf{x}$ and $\mathcal{I}_N \setminus \mathbf{x}$ are transitive, is Inv(σ) for a unique $\sigma \in \mathfrak{S}_N$ (Dushnik and Miller 1941, Guilbaud and Rosenstiehl 1963).

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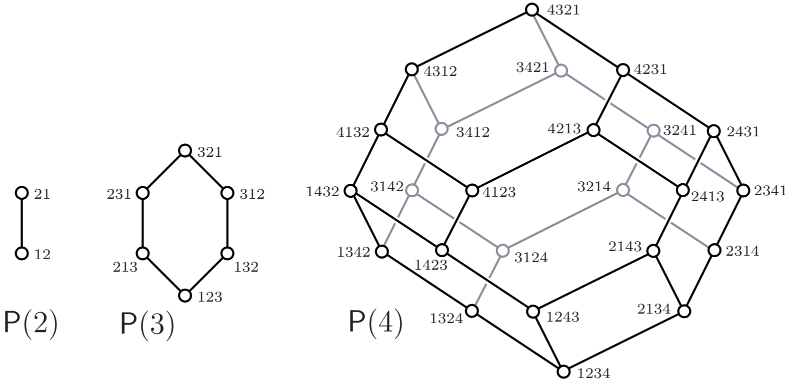
- Observe that each $\mathbf{x} \in \mathrm{P}(N)$ is a strict ordering. It can be proved (Dushnik and Miller 1941) that those are exactly the finite strict orderings of order-dimension 2.

The permutohedra P(2), P(3), and P(4).



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Hence P(N) is an ortholattice.

What makes P(N) a lattice?

- Every $\mathbf{x} \subseteq \mathcal{I}_N$ is contained in a least closed set, namely, $\mathrm{cl}(\mathbf{x}) =$ transitive closure of $\mathbf{x}$:

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- Dually, every $\mathbf{x} \subseteq \mathfrak{I}_N$ contains a largest open set, namely, $\mathrm{int}(\mathbf{x}) = \mathfrak{I}_N \setminus \mathrm{cl}(\mathfrak{I}_N \setminus \mathbf{x})$:

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Theorem (Guilbaud and Rosenstiehl 1963 ?)

$\mathrm{int}(\mathbf{x})$ is closed, for any closed $\mathbf{x} \subseteq \mathcal{I}_N$.

Proof.

Let $(i, j), (j, k) \in \mathrm{int}(\mathbf{x})$ and suppose that $(i, k) \notin \mathrm{int}(\mathbf{x})$.

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- Likewise, $\mathbf{x} \cup \mathbf{y}$ is open, and $\mathbf{x} \vee \mathbf{y} = \mathrm{cl}(\mathbf{x} \cup \mathbf{y})$.

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The permutohedron P(N) is McKenzie-bounded, for every positive integer N.

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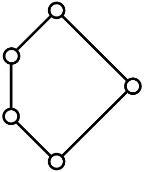
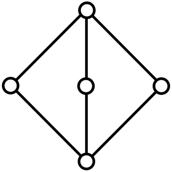
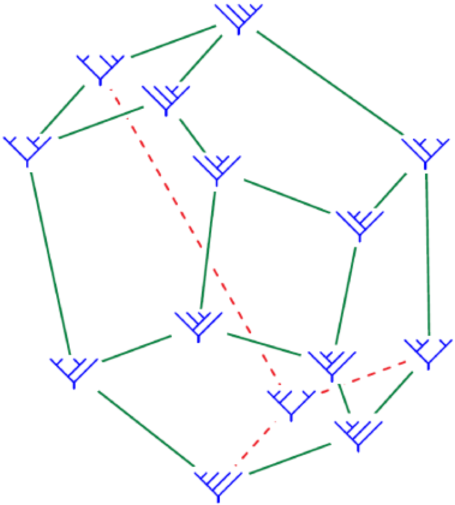




The associahedron, or Stasheff polytope



(from http://www.math.tamu.edu/~jwhite/math613.html)

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Join-irreducibles in $\mathrm{A}_U(N)$ (and $\mathrm{P}(N)$)

- For $(i,j) \in \mathcal{I}_N$, set

$$\langle i,j\rangle_U = \{(x,y) \in \mathcal{I}_N \mid x \in U^{\mathsf{c}} \cup \{i\} \text{ and } y \in U \cup \{j\}\}.$$

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$$\langle i,j\rangle_U = \{(x,y) \in \mathfrak{I}_N \mid x \in U^{\mathsf{c}} \cup \{i\} \text{ and } y \in U \cup \{j\}\}\,.$$

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- The join-irreducible elements of $P(N)$ are the $\langle i,j \rangle_U$, for $(i,j) \in \mathcal{I}_N$ and $U \subseteq [N]$.

OD-graphs of Cambrian lattices

- $\langle x, y\rangle_U \leq \langle z, w\rangle_U$ iff
  - $[x, y] \subseteq [z, w]$,
  - $z < x$ implies $x \notin U$,
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- minimal join-covers are of the form

$$\langle x, y\rangle_U \leq \bigvee\{\langle z_i, z_{i+1}\rangle_U \mid i < n\}$$

where

$$x = z_0 < z_1 < \ldots < z_n = y$$

is a subdivision of the interval $[x, y]$.

All isomorphisms and dual isomorphisms between Cambrians of type A

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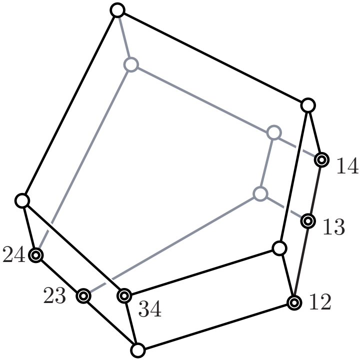
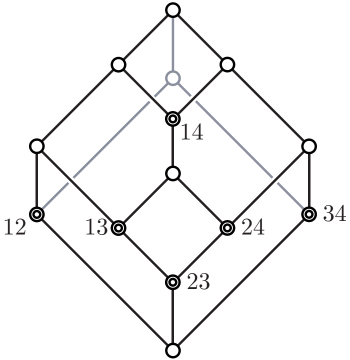
Keep this in mind for Luigi's talk!

Picturing the Cambrian lattices of type A, for $N = 4$





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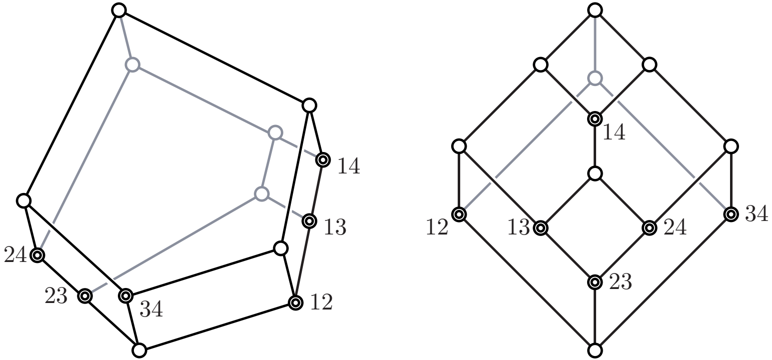

N. Reading observed that each $A_U(N)$ has cardinality $\frac{1}{N+1}\binom{2N}{N}$.

Grätzer's problem for Tamari lattices

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Characterize the (finite) lattices that can be embedded into some Tamari lattice A(N).

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- At that time, no reasonable guess for a solution to Grätzer's problem.
- It is still unknown whether

$$\{L \mid (\exists N)(L \hookrightarrow \mathrm{A}(N))\}$$

is decidable.

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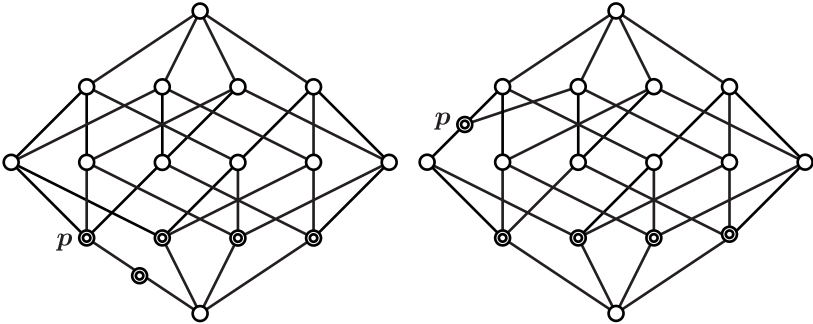
- Conjecture easy to verify for finite distributive lattices.

The lattices B(m, n)



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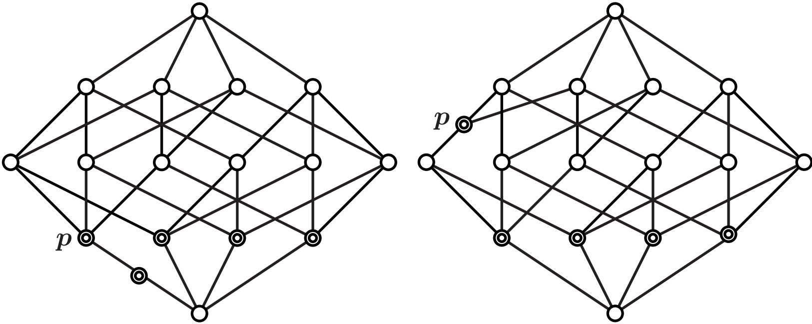

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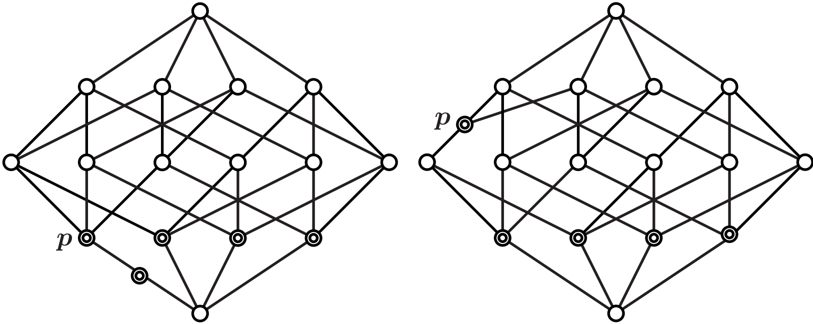

B(1, 3) and B(2, 2), non-atom join-irreducible element is **p**.

- The lattice B(m, n) is defined by doubling the join of m atoms in an $(m + n)$-atom Boolean lattice.
- All lattices B(m, n) are McKenzie-bounded.

B(m, n), A(N), and P(N)

Theorem (S. and W. 2010)

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In particular:

Neither B$(2, 2)$ nor P(4) can be embedded into any A(N) (although they are both McKenzie-bounded).

Vegetables and Gazpachos

- An identity witnessing $\mathrm{B}(2,2) \not\hookrightarrow \mathrm{A}(N)$ is (Veg_1):

$$(\mathrm{a}_1 \vee \mathrm{a}_2 \vee \mathrm{b}_1) \wedge (\mathrm{a}_1 \vee \mathrm{a}_2 \vee \mathrm{b}_2) \leq \bigvee_{i,j \in \{1,2\}} \big((\mathrm{a}_i \vee \tilde{\mathrm{b}}_j) \wedge (\mathrm{a}_1 \vee \mathrm{a}_2 \vee \mathrm{b}_{3-j})\big),$$

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- A most useful tool for proving this is the notion of U-polarized measure, $\mu\colon \mathcal{J}_N \to L$.
- For a finite lattice L, certain U-polarized measures with values in L correspond to lattice embeddings of L into $A_U(N)$.

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$\mathrm{B}(3,3)$ is a homomorphic image of a sublattice of $\mathrm{P}(12)$.

- We prove that a certain $\mathrm{A}_U(12)$ does not satisfy the splitting identity of $\mathrm{B}(3,3)$:

$$\bigwedge_{1\leq j\leq 3} (\mathrm{x}_1 \vee \mathrm{x}_2 \vee \mathrm{x}_3 \vee \mathrm{y}_j) \leq \bigvee_{1\leq i\leq 3} (\hat{\mathrm{x}}_i \wedge \hat{\mathrm{y}}_1 \wedge \hat{\mathrm{y}}_2 \wedge \hat{\mathrm{y}}_3)\,,$$

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- Hence θ_K is the weakest identity failing in K.
- If K is splitting and $K \in \mathrm{HSP}(\mathcal{X})$, then $K \in \mathrm{HSP}(L)$ for some $L \in \mathcal{X}$. (*Proof*: $\mathrm{HSP}(\mathcal{X}) \nsubseteq \mathcal{C}_K$, that is, $\mathcal{X} \nsubseteq \mathcal{C}_K$, so there exists $L \in \mathcal{X}$ with $L \notin \mathcal{C}_K$.)

No separating identity for B(3, 3) (cont'd)

- Relevant values of the x_i, y_i obtained with help of the Prover9-Mace4 program (yields $U = \{5, 6, 9, 10, 11\}$).

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- Variety membership problem, in the $\mathrm{A}_U(N)$, captured by combinatorial objects called scores.
- An (m, n)-score, with respect to $U \subseteq [N]$, expresses a certain tiling property of $m + n$ copies of $[N]$.

Theorem (S. and W. 2014)

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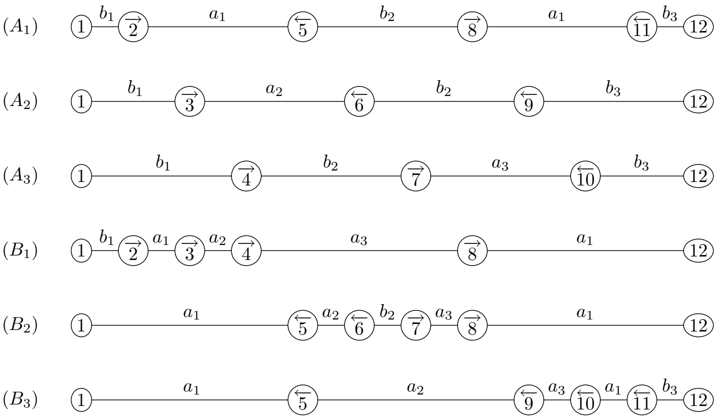
The following statements are equivalent, for all positive integers m, n, N and all $U \subseteq [N]$:

1. $\mathrm{B}(m, n)$ belongs to the lattice variety generated by $\mathrm{A}_U(N)$.
2. $\mathrm{A}_U(N)$ does not satisfy the splitting identity of $\mathrm{B}(m, n)$.
3. There exists an (m, n)-score on $[N]$ with respect to U.

The score for $\mathsf{B}(3,3) \in \mathsf{HS}(\mathsf{A}_U(12))$



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- It is well-known (Day 1977) that every identity satisfied by all finite splitting lattices is trivial.
- Due to the splitting identities, the question above is equivalent to: "Is every finite McKenzie-bounded (resp., splitting) lattice a homomorphic image of a sublattice of some P(N)?"

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- Due to the splitting identities, the question above is equivalent to: "Is every finite McKenzie-bounded (resp., splitting) lattice a homomorphic image of a sublattice of some P(N)?"
- Verified above in the case of B(3, 3) (with P(12)).

Outline

1. Elementary theory of permutohedra

2. An identity satisfied by all the permutohedra
   - EA-duets, sopranos, and bassos
   - Tensor prod
   - Box prod
   - P(N) $\models \theta_L$

3. Decidability of the weak Bruhat ordering on permutations via MSOL and S1S

4. No identities for generalized permutohedra

Constraints in lattice theory

- Two variables: E and A, interpret them in $\mathcal{H}$.
- E and A are both singers.
- E is male and A is female.
- They are close to each other (lovers, possibly?).

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Any solution ?

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"Born in Zell im Wiesental (Baden-Württemberg, Germany), Aloysia Weber (later on Aloysia Weber-Lange) was one of the four daughters of the musical Weber family."

The Bass: Édouard de Reszke (1853 – 1917)

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"A Polish bass from Warsaw. Born with an impressive natural voice and equipped with compelling histrionic skills, he became one of the most illustrious opera singers active in Europe and America during the late-Victorian era."

Galois adjunctions

- A Galois adjunction between posets K and L is a pair (f, h), where $f\colon K \to L$, $h\colon L \to K$, and

$$f(x) \leq y \Leftrightarrow x \leq h(y)\,, \quad \forall (x, y) \in K \times L\,.$$

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- f is the lower adjoint and h is the upper adjoint.
- f is a join-homomorphism and h is a meet-homomorphism.
- Each one of f and h determines the other.

EA-duets

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Relations from the adjunction.

$$f(x) \leq y \Leftrightarrow x \leq h(y)\,,$$
$$y \leq g(x) \Leftrightarrow h(y) \leq x\,,$$
$$f(x) = \text{least element of } h^{-1}\{x\}\,,$$
$$g(x) = \text{largest element of } h^{-1}\{x\}\,.$$

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  f is the lower adjoint of h and g is the upper adjoint of h.

Remark !!! In categorical logic, we would write

$$f := \exists_h \dashv h \dashv \forall_h =: g\,.$$

Whence, EA-duet.

Lemma

Let $f, g \colon K \to L$. Then (f, g) is an EA-duet iff f is a join-homomorphism, g is a meet-homomorphism, and

$$f(x) \le g(y) \Leftrightarrow x \le y\,, \quad \forall (x, y) \in K \times K\,.$$

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$$f(x) \leq g(y) \Leftrightarrow x \leq y\,, \quad \forall (x, y) \in K \times K\,.$$

Necessarily,

$$H = \bigcup_{x \in K} [f(x), g(x)]\,,$$

$$h(y) = \text{unique } x \in K \text{ such that } f(x) \leq y \leq g(x)\,.$$

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Lemma

Let (u, v) be a prime critical pair of K.
A pair $f, g \colon K \to L$ is an EA-duet iff

- *f is a join-homomorphism, g is a meet-homomorphism,*
- *$f \leq g$, and $f(u) \not\leq g(v)$.*

Proof.

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Since $x \wedge y < x$, the congruence $\mathrm{con}(x \wedge y, x)$ is nonzero, thus it contains the monolith $\mathrm{con}(u \wedge v, u)$ of K.

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Since $u \wedge v$ is a lower cover of u, the weak projectivity $[x \wedge y, x] \Rightarrow [u \wedge v, u]$ holds.

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Since $u \wedge v$ is a lower cover of u, the weak projectivity $[x \wedge y, x] \Rightarrow [u \wedge v, u]$ holds.
Since $f(u) \nleq g(u \wedge v)$, it follows that $f(x) \nleq g(x \wedge y)$, a contradiction. □

Use:

Lemma

If $[a, b]$ weakly transposes to $[c, d]$, then
$$f(b) \leq g(a) \text{ implies } f(d) \leq g(c).$$

For example, if $d = b \vee c$ and $a \leq c$, then

$$f(d) = f(b) \vee f(c) \leq g(a) \vee g(c) \leq g(c)\,.$$

Bringing Édouard and Aloysia tighter together

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- Necessarily, f is a join-homomorphism, g is a meet-homomorphism, and $f \leq g$.
- $f \colon K \to L$ is a lattice homomorphism iff (f, f) is tight.

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A nonzero element $p \in L$ is join-prime in L if $p \leq x \vee y$ implies that either $p \leq x$ or $p \leq y$, $\forall x, y \in L$.

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Lemma

For lattices K and L of finite length, let (f, g) be a tight EA-duet on (K, L). Then f and g agree on 0_K, 1_K, all join-primes, and all meet-primes of K.

Proof.

Prove that whenever p is join-prime and g is isotone, $g(p) = g^{\vee}(p)$.

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Prove that whenever p is join-prime and g is isotone, $g(p) = g^{\vee}(p)$.

The map $g' \colon K \to L$ defined by

$$g'(x) = \begin{cases} g(p)\,, & \text{if } p \leq x\,, \\ g(0_K)\,, & \text{otherwise} \end{cases}$$

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Now $g(p) = g'(p) \leq g^{\vee}(p) \leq g(p)$. □

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Similar for $g(0) = g^{\vee}(0)$.

□

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...use congruence distributivity and Jónsson Lemma. □

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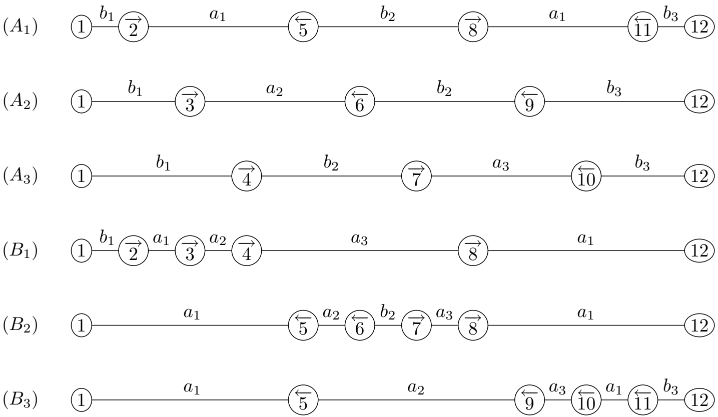
- Hence θ_K is the weakest identity failing in K.
- If K is splitting and $K \in \mathrm{HSP}(\mathcal{X})$, then $K \in \mathrm{HSP}(L)$ for some $L \in \mathcal{X}$. (*Proof*: $\mathrm{HSP}(\mathcal{X}) \nsubseteq \mathcal{C}_K$, that is, $\mathcal{X} \nsubseteq \mathcal{C}_K$, so there exists $L \in \mathcal{X}$ with $L \notin \mathcal{C}_K$.)

Scores again



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- We tried to build (n, m)-scores, for $n \geq 3$ and $m \geq 3$ and $n + m \geq 7$.

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- We tried to build (n, m)-scores, for $n \geq 3$ and $m \geq 3$ and $n + m \geq 7$.
- We could not disprove existence of such a score.

- Needed some different ideas, and to step away from the $\mathrm{B}(n, m)$.

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- The bi-ideals form an algebraic lattice.
- $A \otimes B = (\vee, 0)$-semilattice of all compact bi-ideals of $A \times B$.

Useful bi-ideals, universal property

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- Pure tensors:

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- Boxes:

$$a \,\square\, b = \{(x, y) \mid x \leq a \text{ or } y \leq b\} .$$

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- The map $(a, b) \mapsto a \otimes b$ is the universal bimorphism on $A \times B$.

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- For finite lattices this does not matter.

Tensor products and congruences

Theorem (Grätzer and W. 1999)

Let C be a capped sub-tensor product of $A \otimes B$. Then

$$\mathrm{Con_c}\, C \cong (\mathrm{Con_c}\, A) \otimes (\mathrm{Con_c}\, B)\,,$$

where $\mathrm{Con_c}\, L$ denotes the $(\vee, 0)$-semilattice of all compact congruences of L.

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- Thus we need another construction.

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Lemma

Let A and B be finite lattices. If A and B are both McKenzie-bounded (resp., splitting), then so is $A \mathbin{\Box} B$.

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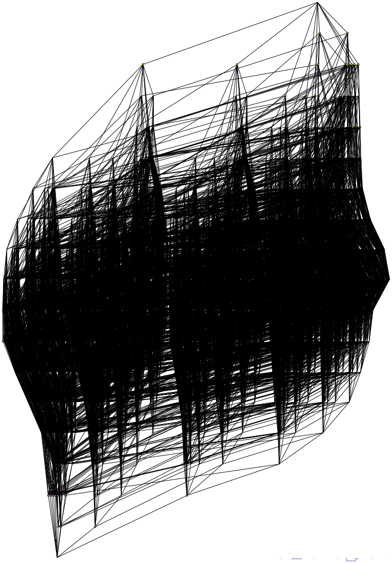
- Brute force computation shows that it has 3,338 elements.

A portrait view of $N_5 \square B(3,2)$



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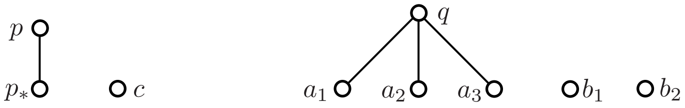
- Label the join-irreducible elements of N_5 and $B(3,2)$ as on the following picture.



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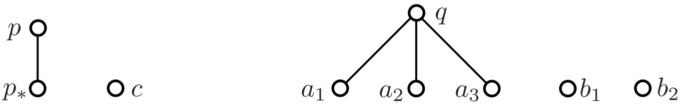

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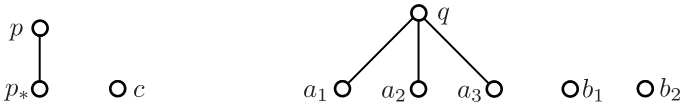

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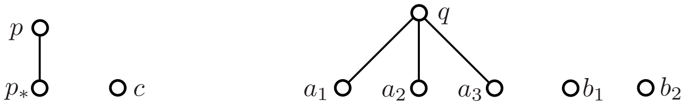

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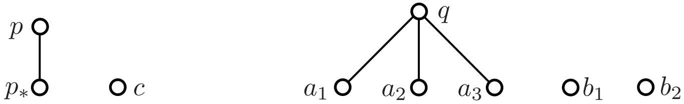

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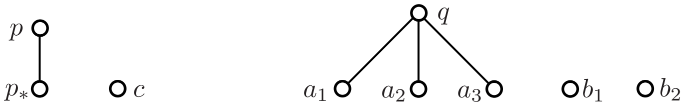

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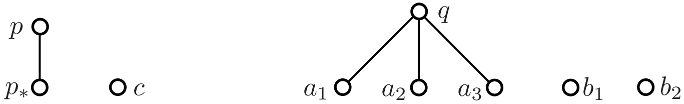

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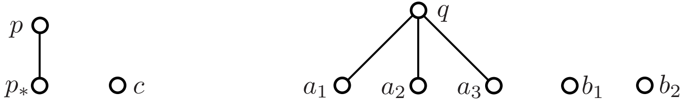

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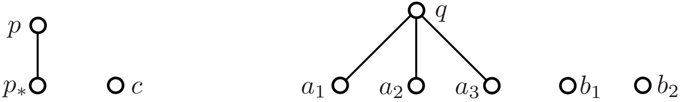

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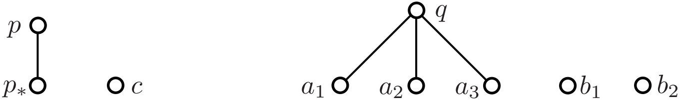

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- We have thus obtained that

$$(1, N) \in f(p \otimes q) \setminus g(p_* \Box q_*) .$$

A crucial lemma

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$\langle 1, N\rangle_U \subseteq f(p \otimes q)$, thus

$$\begin{aligned} \langle 1, N\rangle_U \cap f(c \otimes q) &\subseteq f(p \otimes q) \wedge f(c \otimes q) \\ &\subseteq g(p \otimes q) \wedge g(c \otimes q) \\ &= g\big((p \otimes q) \wedge (c \otimes q)\big) \\ &= g\big((p \wedge c) \otimes q\big) \\ &= g(0)\,. \end{aligned}$$

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Since $c \otimes q \not\leq c \otimes q_*$, we get a contradiction.

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- Hence there are subdivisions

$$x = z_0^j < \cdots < z_{n_j}^j = y\,,$$

where $j \in \{1, 2\}$, where each $(z_i^j, z_{i+1}^j) \in f(c \otimes d_i^j)$ for some $d_i^j \in \{a_1, a_2, a_3, b_j\}$.

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- Now $d \in \{a_1, a_2, a_3, b_j\}$ thus $d \wedge q \leq q_*$, so $(x, y) \in g(c \otimes q_*)$ and we are done.
- We may thus assume that $n_j > 1$ $\forall j \in \{1, 2\}$.

The following claim expresses a crucial pattern of the finite sequences $(z_i^j)_{0 \leq i \leq n_j}$, with respect to belonging to U.

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- Suppose otherwise, with (say) $i > 0$.
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- Let $d \in \{a_1, a_2, a_3, b_j\}$ such that $(z_{i-1}^j, z_i^j) \in f(c \otimes d)$.
- Since $(z_i^j, z_{i+1}^j) \in \langle 1, N\rangle_U$ (by assumption) and $(x, y) \in f(c \otimes d)$, we get $(z_i^j, z_{i+1}^j) \in g(0) \subseteq g(c \otimes d)$ by the Lemma.

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- Now c and d are both join-prime, thus so is $c \otimes d$.

The following claim expresses a crucial pattern of the finite sequences $(z_i^j)_{0 \leq i \leq n_j}$, with respect to belonging to U.

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There is no $i < n_j$ such that $z_i^j \notin U$ and $z_{i+1}^j \in U$.

- Suppose otherwise, with (say) $i > 0$.
- Let $d \in \{a_1, a_2, a_3, b_j\}$ such that $(z_{i-1}^j, z_i^j) \in f(c \otimes d)$.
- Since $(z_i^j, z_{i+1}^j) \in \langle 1, N \rangle_U$ (by assumption) and $(x, y) \in f(c \otimes d)$, we get $(z_i^j, z_{i+1}^j) \in g(0) \subseteq g(c \otimes d)$ by the Lemma.
- Now c and d are both join-prime, thus so is $c \otimes d$.
- Since (f, g) is a tight EA-duet and by "agreement on basic things", $(z_i^j, z_{i+1}^j) \in f(c \otimes d)$.

- Since $(z^j_{i-1}, z^j_i) \in f(c \otimes d)$ as well, we get $(z^j_{i-1}, z^j_{i+1}) \in f(c \otimes d)$.

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- In contradiction with the minimality of n_j.

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- Thus for each $j \in \{1, 2\}$, there exists a unique $m_j \in [0, n_j - 1]$ such that

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$$\begin{aligned} z^j_i &\in U && \text{whenever } 0 < i \leq m_j\,, \\ z^j_i &\notin U && \text{whenever } m_{j+1} < i < n_j\,. \end{aligned}$$

- Since $(z^j_{i-1}, z^j_i) \in f(c \otimes d)$ as well, we get $(z^j_{i-1}, z^j_{i+1}) \in f(c \otimes d)$.
- In contradiction with the minimality of n_j.
- Thus for each $j \in \{1, 2\}$, there exists a unique $m_j \in [0, n_j - 1]$ such that

$$\begin{aligned} z^j_i &\in U && \text{whenever } 0 < i \leq m_j\,, \\ z^j_i &\notin U && \text{whenever } m_{j+1} < i < n_j\,. \end{aligned}$$

- From then on, the proof becomes quite complicated, comparing the positions of the z^j_i, using repeatedly "agreement on basic things", and calculating various joins in $N_5 \square B(3, 2)$.

A zoo of cases

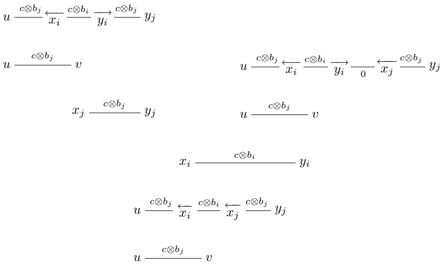

FIGURE 10.1. Cases 1.a (up-left), 1.b (up-right), and 2 (down) in the proof of $(u, v) \in f(c \otimes a_k) \cup \Delta$ in Claim 3

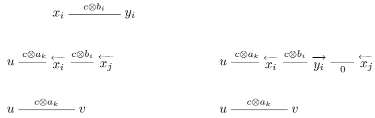

FIGURE 10.2. Cases 1 (left) and 2 (right) in the proof of $v = x_j$ in Claim 3

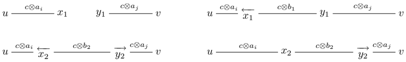

FIGURE 10.3. Final cases in the proof of Lemma 10.3: Case 1 (left) and Case 2 (right)

Outline

Permutohedra as lattices

Theorem (Guilbaud et Rosenstiehl, 1963)

The permutohedra P(N) (with the weak Bruhat order) are lattices.

Permutohedra as lattices

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The permutohedra P(N) (with the weak Bruhat order) are lattices.

State of art before (W. and S. 2014, ...):

Theorem (Claude Le Conte de Poly-Barbut, 1994)

The permutohedra P(N) are semi-distributive.

Theorem (Nathalie Caspard, 1999)

The permutohedra P(N) are McKenzie–bounded.

The equational theory of permutohedra

The word problem for permutohedra

Given lattice terms s and t, does the equality

$$\mathrm{P}(N) \models s = t\,,$$

hold, for each $N \geq 1$?

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Theorem (W. and S. 2014)

The word problem for permutohedra is decidable.

The Cambrian lattice $A_U(N)$

Fix N and $U \subseteq \{1, \dots, N\}$.

Definition

A subset $X \subseteq \{(i,j) \mid 1 \leq i < j \leq N\}$ is U-closed if

1. it is transitive : $(i,j), (j,k) \in X$ implies $(i,k) \in X$;
2. if $i < j < k$ and $(i,k) \in X$, then
   - $j \in U$ implies $(i,j) \in X$,
   - $j \notin U$ implies $(j,k) \in X$.

Let :

$$A_U(N) := \{X \subseteq \mathcal{I}_N \mid X \text{ est } U\text{-closed}\} .$$

Proposition

$A_U(N)$, with subset inclusion, is a lattice.

Pemutohedra and Cambrians lattices

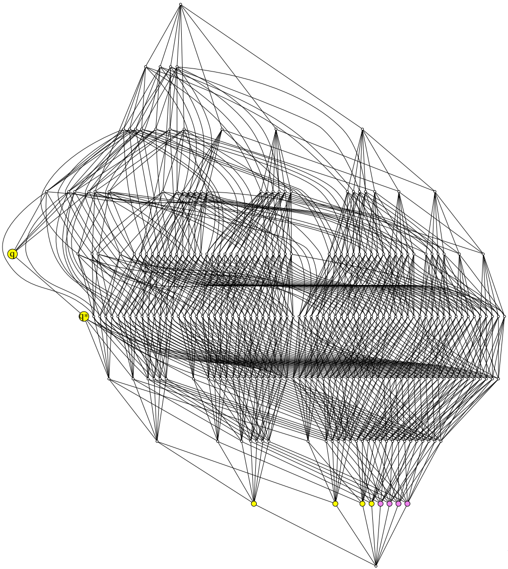
Proposition

For all pair of lattice terms s, t, we have

$$\mathrm{P}(N) \models s = t \text{ for all } N$$
$$\text{iff}$$
$$\mathrm{A}_U(N) \models s = t \text{ for all } N \text{ and } U \subseteq [1, \ldots, N]\,.$$

As the Cambrian lattices are the subdirectly irreducible quotients of the permutohedra.

The lattice B(4, 4)



The lattices B(n, m)

The lattice B(n, m) is obtained from a Boolean-algebra over $n + m$ atoms, by doubling the join of n-atoms. Let HSP(P) be the variety generated by the Permutohedra.

The lattices $\mathrm{B}(n, m)$

The lattice $\mathrm{B}(n, m)$ is obtained from a Boolean-algebra over $n + m$ atoms, by doubling the join of n-atoms. Let $\mathrm{HSP}(\mathrm{P})$ be the variety generated by the Permutohedra.

Problem

Given n and m, does the lattice $\mathrm{B}(n, m)$ belong to $\mathrm{HSP}(\mathrm{P})$?

EA-duets and scores

Proposition

TFAE:

1. $\mathrm{B}(n, m) \in \mathrm{HSP}(\mathrm{P}(N) \mid N \geq 1)$,

EA-duets and scores

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EA-duets and scores

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3. $\exists N, U$ s.t. $\mathrm{B}(n, m) \in \mathrm{HS}(\mathrm{A}_U(N))$,
4. $\exists N, U$ and an EA-duet $(f, g) : \mathrm{B}(n, m) \longrightarrow \mathrm{A}_U(N)$,

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3. $\exists N, U$ s.t. $B(n, m) \in HS(A_U(N))$,
4. $\exists N, U$ and an EA-duet $(f, g) : B(n, m) \longrightarrow A_U(N)$,
5. $\exists N, U$ and an *(n, m, N, U)-score*.

EA-duets and scores

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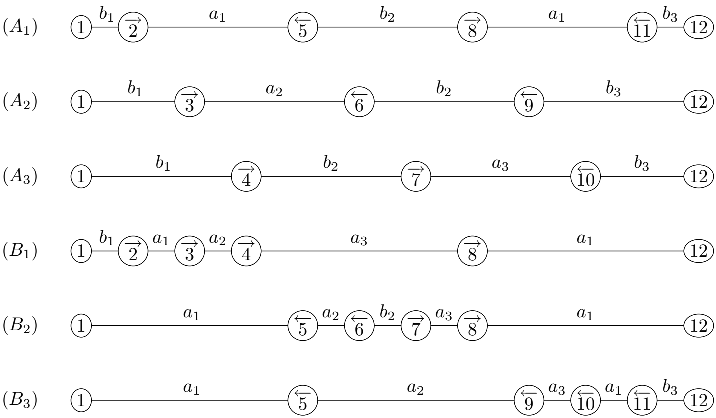
5. $\exists N, U$ and an *(n,m,N,U)-score*.

Recall: an EA-duet $(f,g) : \mathrm{B}(n,m) \longrightarrow \mathrm{A}_U(N)$ is such that

1. f is a $\vee$-homomorphism,
2. g is a $\wedge$-homomorphism,
3. $f(x) \leq g(y)$ iff $x \leq y$.

How does an (n, m, N, U)-score look like?

A $(3, 3, 12, \{5, 6, 9, 10, 11\})$-score :



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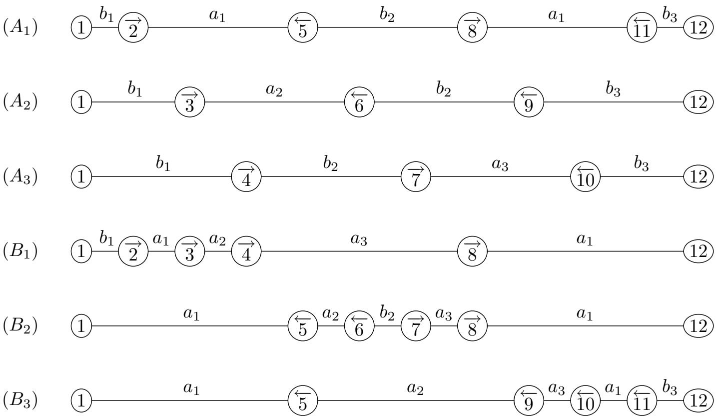

(therefore $\mathrm{B}(3,3) \in \mathrm{HSP}(\mathrm{P}(N) \mid N \geq 1)$).

Def. of an (n, m, N, U)-score

Basso parts: for $i = 1, \ldots, n$,

- a subdivision of the interval $[1, N]$,
- each interval of the subdivision being labeled by a_j, $j = 1, \ldots, m$, or b_i;

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Scores from EA-duets

- The half-score of Bassos codes the join-homorphism

$$\mathsf{B}(n, m) \xrightarrow{f} \mathsf{A}_U(N)\,.$$

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(whence the meet-homomorphism $g : \mathrm{B}(n, m) \longrightarrow \mathrm{A}_U(N)$).

- Solos and consonances contraints translate the relations $f \leq g$ and $f(q) \not\leq g(q_*)$.

Half scores code join-homomorphisms

By the example.

- Minimal non-trivial join-cover relations in B(3, 3).

$$q \leq \bigvee \vee a_1 \vee a_2 \vee a_3 \vee b_j\,, \qquad j = 1, 2, 3\,.$$

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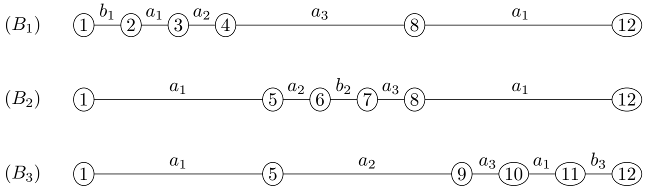
- These relations represented as subdivisions:



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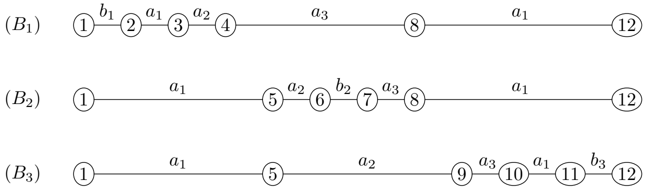

- Define then:

$$f(x) := \bigvee \{\langle i, j \rangle_U \mid (i, j) \text{ is labeled by } x\}\,, \qquad \text{for } x \text{ and atom}\,,$$

$$f(q) := \langle 1, N \rangle_U \vee \bigvee \{f(x) \mid x \leq q, x \in \mathrm{Ji}(\mathrm{B}(3,3))\}\,.$$

Back to def of an (n, m, N, U)-score

Basso parts: for $i = 1, \dots, n$,

- a subdivision of the interval $[1, N]$,
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Alto parts: for $j = 1, \dots, m$,

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Back to def of an (n, m, N, U)-score

Basso parts: for $i = 1, \ldots, n$,

- $B_i \subseteq [1, N]$,
- $x, y \in B_i$ with $succ(x, y, B_i)$ implies $\bigvee_{j=1,\ldots,m} B_{i,a_j}(x, y) \vee B_{i,b_i}(x, y)$;

Alto parts: for $j = 1, \ldots, m$,

- $A_j \subseteq [1, N]$,
- $x, y \in A_j$ with $succ(x, y, A_j)$ implies $\bigvee_{i=1,\ldots,n} A_{j,b_i}(x, y) \vee A_{j,a_j}(x, y)$;

Solos: every Basso peak is a b_i ; every Soprano valley is an a_j ;

Consonances: for each (x, y) of some basso, (z, w) of some soprano, if $(x, y) \not\perp_U (z, w)$, then the label of (x, y) is equal to the label of (z, w).

Summarizing

- We can represent a (n, m, N, U)-score via subsets

$$B_i\,, A_j\,,\ B_{i,\sigma}\,,\ A_{j,\sigma}\,,$$
$$\text{where } i = 1, \ldots m,\ j = 1, \ldots n\,,\ \sigma \in \{a_1, \ldots, a_n, b_1, \ldots, b_m\},$$

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- We can suppose that $B_i \, , A_j \, , \, B_{i,\sigma} \, , \, A_{j,\sigma}$ are all subsets of integers (that is unary predicates or monadic) ;

- The property

  $$B_i, A_j, B_{i,\sigma}, A_{j,\sigma} \text{ is an } (n, m, N, U)\text{-score}$$

  is definable in $MSOL(succ)$ (monadic second order logic of one successor).

$MSOL(succ)$, $S1S$, and Büchi's Theorem

- $MSOL(succ)$: logic formulas built via the grammar

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Corollary

The problem $\mathrm{B}(n, m) \in \mathrm{HSP}(\mathrm{P}(N) \mid N \geq 1)$ is decidable.

Generalizing to splitting lattices

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- ... and represent iterated scores within $MSOL(succ)$.

Generalizing to equations

- By R. McKenzie theory, splitting lattices are almost failure of equations. For example:

$$N_5 \in \mathrm{HSP}(K) \qquad \text{iff} \qquad K \not\models \text{modular equation}\,.$$

- A lattice term is naturally structured in join-covers. For example, for $t := x \wedge (y \vee z)$, we have:

$$t \leftarrow x\,, \qquad t \leftarrow \{y, z\}\,.$$

Scores for a pair of terms

Given two terms s, t, we can define (within MSOL(succ)) the notion of $(s, t, N, U,)$-score, so that:

Proposition

TFAE:

1. $\mathrm{P} \not\models s \leq t$;
2. $\exists N, U$ s.t. $\mathrm{A}_U(N) \not\models t \leq s$;
3. $\exists N, U, v : vars(s,t) \longrightarrow \mathrm{A}_U(N)$ s.t. $s(v) \not\leq t(v)$;
4. $\exists N', U'$ and an (t, s, N', U')-score.

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Theorem

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- Why $\mathrm{B}(3,2) \in \mathrm{HSP}(\mathrm{P}(N) \mid N \geq 1)$, while $\mathrm{N}_5 \,\square\, \mathrm{B}(3,2) \notin \mathrm{HSP}(\mathrm{P}(N) \mid N \geq 1)$?

Outline

Basic definitions

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- Then we set

$$\mathrm{P}(E) \underset{\text{def.}}{=} \{\mathbf{a} \subseteq \boldsymbol{\delta}_E \mid \mathbf{a} \text{ is clopen}\}\,, \quad \text{(that's our guy)}$$

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- Obviously, $\mathrm{P}([1, N]) = \mathrm{P}(N)$!!!

Is P(E) a lattice?

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The following statements hold, for any poset E.

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2. P(E) $=$ P*(E) iff E is crown-free.

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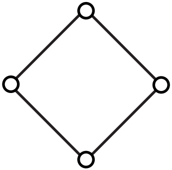
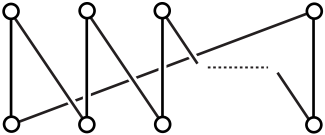
Illustrating square and crowns:





What about McKenzie-boundedness?

Theorem (Caspard, S., and W. 2011)

Let E be a square-free poset. Then the lattice P(E) is a subdirect product of the P(C), for all maximal chains C of E.

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- "Square-free" is just put there in order to ensure that P(E) be a lattice.
- For E an infinite chain, P(E) is not even semidistributive.

Why is $\mathrm{P}^*(E)$ sometimes better than $\mathrm{P}(E)$?

Theorem (Pouzet, Reuter, Rival, and Zaguia 1995)

Let E be a finite poset. Then the inclusion mapping from $\mathrm{P}^*(E)$ into the powerset of $\boldsymbol{\delta}_E$ is cover-preserving.

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There is a finite poset E such that the inclusion mapping from $P(E)$ into the powerset of $\boldsymbol{\delta}_E$ is not height-preserving (thus also not cover-preserving).

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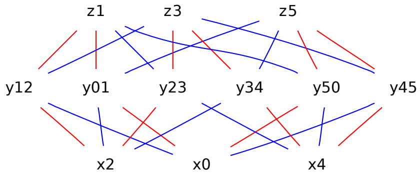
Here is the counterexample:



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- Is there anything left in case E is not square-free?
- It turns out that yes.

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Operators cl and int defined as before:

- $\mathrm{cl}(\mathbf{x})$ is the transitive closure of $\mathbf{x}$,
- $\mathrm{int}(\mathbf{x}) = \mathbf{e} \setminus \mathrm{cl}(\mathbf{e} \setminus \mathbf{x})$.

The lattices Reg($\mathbf{e}$) and $\text{Reg}_{\text{op}}(\mathbf{e})$

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- $\mathbf{x} \mapsto \mathbf{x}^{\perp} = \mathrm{cl}(\mathbf{x}^{\mathsf{c}})$ defines an orthocomplementation on $\mathrm{Reg}(\mathbf{e})$.

The lattices (cont'd)

- $\text{int} \circ \text{cl}$: closure operator on open sets,
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$\mathrm{Clop}(\mathbf{e})$ is an orthocomplemented poset.

It may not be a lattice (e.g., $\mathrm{P}(E) = \mathrm{Clop}(\boldsymbol{\delta}_E)$, for any poset E; take E non square-free).

Some notation

For a transitive relation **e** on a set E, write

$$x \vartriangleleft_{\mathbf{e}} y \underset{\text{def.}}{\Longleftrightarrow} (x, y) \in \mathbf{e}\,,$$

$$x \trianglelefteq_{\mathbf{e}} y \underset{\text{def.}}{\Longleftrightarrow} (\text{either } x \vartriangleleft_{\mathbf{e}} y \text{ or } x = y)\,,$$

for all $x, y \in E$.

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As $a \lhd_{\mathbf{e}} a$ may occur, a may belong to $]a, b]_{\mathbf{e}}$.

Square-free transitive relations

Definition

A transitive relation **e** on a set E is square-free if the preordered set $(E, \trianglelefteq_{\mathbf{e}})$ is square-free. That is,

$$(\forall a, b, x, y)\Big(\big(a \trianglelefteq_{\mathbf{e}} x \text{ and } a \trianglelefteq_{\mathbf{e}} y \text{ and } x \trianglelefteq_{\mathbf{e}} b \text{ and } y \trianglelefteq_{\mathbf{e}} b\big) \Longrightarrow (\text{either } x \trianglelefteq_{\mathbf{e}} y \text{ or } y \trianglelefteq_{\mathbf{e}} x)\Big)\,.$$

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Theorem (S. and W. 2012)

The following are equivalent, for any transitive relation $\mathbf{e}$:

1. Clop($\mathbf{e}$) is a lattice.
2. Clop($\mathbf{e}$) $=$ Reg($\mathbf{e}$).
3. int($\mathbf{x}$) is closed, for any closed $\mathbf{x} \subseteq \mathbf{e}$.
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Permutohedra on non square-free posets

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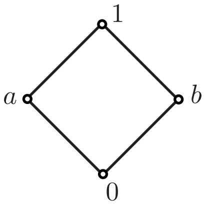
The extended permutohedron on the square B_2

There it goes:



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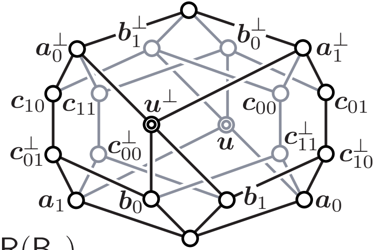

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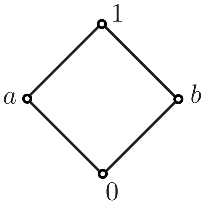

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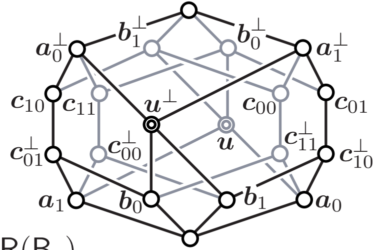

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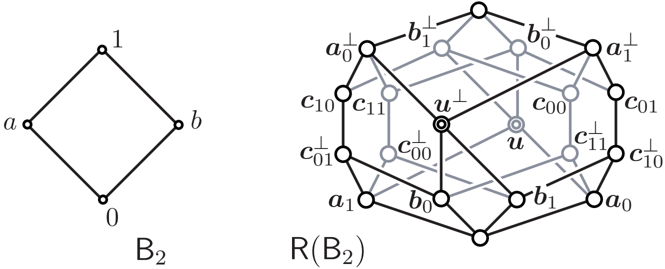

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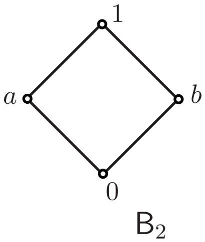

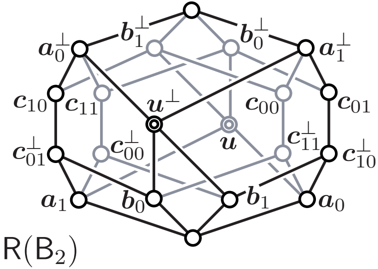

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Bip(N): basic observations

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- $\operatorname{card}\mathrm{Bip}(2) = 10$, $\operatorname{card}\mathrm{Bip}(3) = 74$, $\operatorname{card}\mathrm{Bip}(4) = 730$, $\operatorname{card}\mathrm{Bip}(5) = 9{,}002$.

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- Each $\mathrm{Bip}(N)$ is a graded lattice (Hetyei and Krattenthaler 2011).

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Example ($N = 5$):

$$Part(\mathbf{x}) := (\{2,3\}, \overline{\{4\}}, \overline{\{1,5\}})$$

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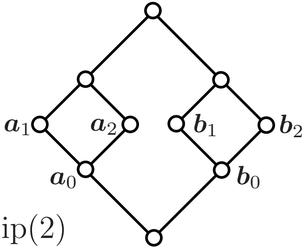

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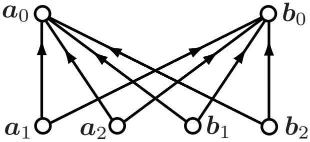

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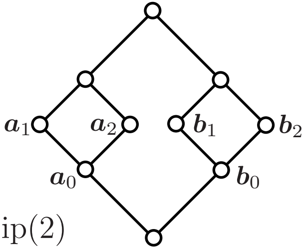

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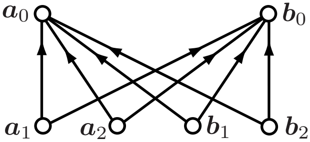

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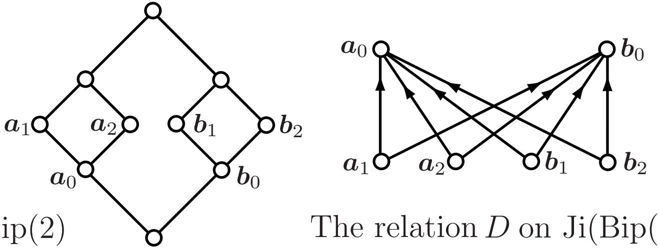
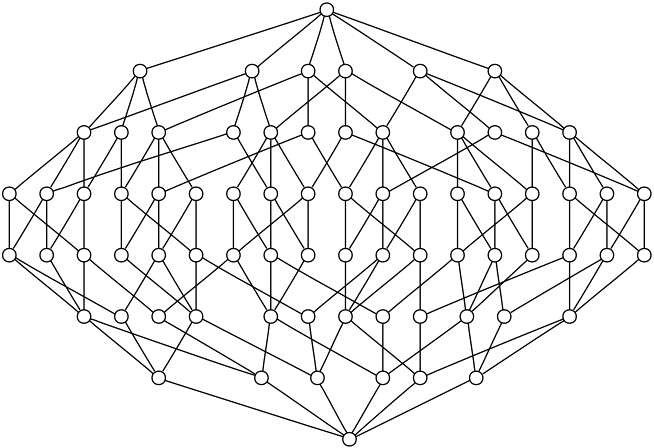


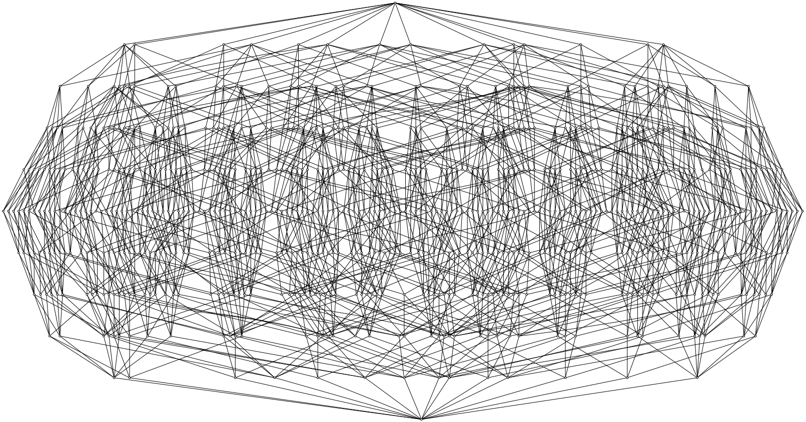
Bip(2) — The relation D on Ji(Bip(2))

- In particular, Bip(2) is McKenzie-bounded.
- This does not extend to higher bipartition lattices: for example, M$_3$ embeds into Bip(3), so Bip(3) is not even semidistributive.

The lattice Bip(3)



The lattice Bip(4)



Some open problems

Problem (S. and W. 2012)

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Is there a nontrivial lattice (ortholattice) identity satisfied by every Bip(N)?

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- We denote by $\mathcal{F}(\mathbf{e})$ the set of all triples (a, b, U), where $(a, b) \in \mathbf{e}$, $U \subseteq [a, b]_{\mathbf{e}}$, and $a \neq b$ implies that $a \notin U$ and $b \in U$.

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- Observe that $\langle a, b; U\rangle$ is bipartite (i.e., it cannot have both (x, y) and (y, z)) iff $a \neq b$. If $a = b$, say that $\langle a, b; U\rangle$ is a clepsydra.

Recognizing the completely join-irreducible elements in Reg($\mathbf{e}$)

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The following statements hold, for any transitive relation $\mathbf{e}$.

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The join-dependency relation on Reg(**e**), the antisymmetric case

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Let **e** be a finite, antisymmetric, transitive relation and let $\mathbf{p}_i = \langle a_i, b_i; U_i \rangle$ be completely join-irreducible in Reg(**e**), for $i \in \{0, 1\}$. Then $\mathbf{p}_0 \; D \; \mathbf{p}_1$ in Reg(**e**) iff

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Hence, if Reg(**e**) is McKenzie-bounded, then it is a direct product of extended permutohedra on finite posets and copies of $\{0, 1\}$ and Bip(2).

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Case of finite chains: solved above (all P(N) $\models \theta_{N_5 \square B(3,2)}$).

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Minimal subdirect decomposition of Bip(N)

- $a \in [N]$ is isolated in $\mathbf{x} \in \mathrm{Bip}(N)$ if $((i,a) \in \mathbf{x}$ and $(a,i) \in \mathbf{x}) \Leftrightarrow i = a$, $\forall i \in [N]$.

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- $\mathrm{S}(N,k)$ is a self-dual lattice (not necessarily a sublattice of Bip(N)), and $\mathrm{S}(N,k) \cong \mathrm{S}(N, N-1-k)$ (so it suffices to consider $0 \leq 2k < N$).

Minimal subdirect decomposition of Bip(N)

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Bip(N) is a subdirect product of copies of the $\mathrm{S}(N,k)$ (minimal subdirect decomposition).
If $N \geq 3$, then $\mathrm{S}(N,k) \not\hookrightarrow \mathrm{Bip}(N)$.

The bip-Cambrian lattices $\mathrm{S}(N, k)$

- Cardinalities for small values: $\operatorname{card} \mathrm{S}(3,0) = 24$, $\operatorname{card} \mathrm{S}(3,1) = 21$; $\operatorname{card} \mathrm{S}(4,0) = 158$, $\operatorname{card} \mathrm{S}(4,1) = 142$; $\operatorname{card} \mathrm{S}(5,0) = 1{,}320$, $\operatorname{card} \mathrm{S}(5,1) = 1{,}202$, $\operatorname{card} \mathrm{S}(5,2) = 1{,}198$.

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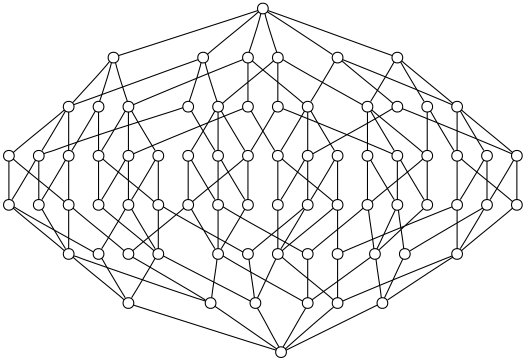
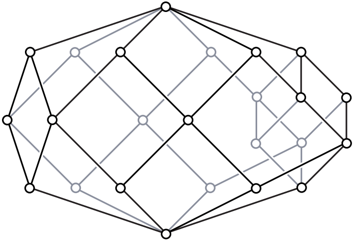
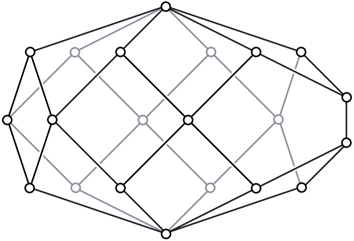
- Recall the picture of Bip(3):



Pictures of $S(3,0)$ and $S(3,1)$





The congruence lattice of Bip(N)

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Let $\mathbf{p}$ and $\mathbf{q}$ be join-irreducible elements in Bip(N), where $N \geq 3$. Then $\mathrm{con}(\mathbf{p}_*, \mathbf{p}) \subseteq \mathrm{con}(\mathbf{q}_*, \mathbf{q})$ iff

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Corollary (S. and W. 2012)

Let $N \geq 3$. Then the congruence lattice of Bip(N) is Boolean on $N \cdot 2^{N-1}$ atoms, with a top element added.

No identities in all R(E)

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- A solution of that refinement problem is a subdivision

$$x = z_0 < z_1 < \cdots < z_n = y$$

  such that $\mu(z_i, z_{i+1}) \leq a_{j_i}$, for each $i < n$.

- Given a polarized measure $\mu : \boldsymbol{\delta}_E \to L$, the map $\varphi\colon L \to \mathrm{R}(E)$ defined by

$$\varphi(a) = \{(x, y) \in \boldsymbol{\delta}_E \mid \mu(x, y) \leq a\}, \quad \forall a \in L,$$

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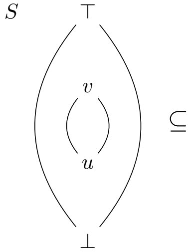
- If every refinement problem has a solution, then φ is a join-homomorphism.

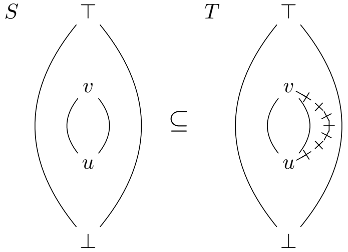
Proof of "no identities in all R(E)"

- Given a finite meet-semidistributive lattice L, a finite dismantlable lattice S, a polarized measure $\mu\colon \boldsymbol{\delta}_S \to L$, and a refinement problem $\mu(u, v) \leq a_0 \vee a_1$,



Proof of "no identities in all R(E)"

- Given a finite meet-semidistributive lattice L, a finite dismantlable lattice S, a polarized measure $\mu\colon \boldsymbol{\delta}_S \to L$, and a refinement problem $\mu(u, v) \leq a_0 \vee a_1$, we find a finite dismantlable lattice T extending S and a polarized measure $\nu\colon \boldsymbol{\delta}_T \to L$ extending μ, such that the refinement problem $\nu(u, v) \leq a_0 \vee a_1$ has a solution in T.



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- Define a map $f\colon (S \downarrow u) \times \omega \to L$ inductively, by

$$\begin{aligned} f(x,0) &= \mu(x,u)\,, \\ f(x,k+1) &= \bigwedge\bigl(\mu(x,t) \vee f(t,k+1)\bigr) \\ &\qquad\qquad \wedge \bigl(f(x,k) \vee a_{\varepsilon(k)}\bigr) \wedge \mu(x,v)\,, \end{aligned}$$

where the $\bigwedge$ is taken over all t with $x < t \leq u$.

Proof (cont'd II)

- Then $f(x,k) \leq f(x,k+1)$, thus, since L is finite, there exists m such that $f(x,k) = f(x,m)$ $\forall x \leq u$ and $\forall k \geq m$.

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- Then T and ν are as required.

Proof (end)

- By starting with μ_0 surjective (e.g., $E_0 = L \setminus \{0\}$, $\mu_0(x, y) = x$), and by repeating the process countably many times, we reach a surjective polarized measure $\mu\colon \boldsymbol{\delta}_E \to L$, where E is a countable union of finite dismantlable lattices, for which every refinement problem has a solution.

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- Extension of the latter result to square-free case hopeless, because in that case, R(E) = P(E) is a subdirect product of P(N)s.

Regular closed sets

- Closure space: pair (Ω, φ), where $\varphi \colon \mathfrak{P}(\Omega) \to \mathfrak{P}(\Omega)$, with $\varphi(\varnothing) = \varnothing$, $X \subseteq Y \Rightarrow \varphi(X) \subseteq \varphi(Y)$, $X \subseteq \varphi(X)$, $\varphi \circ \varphi = \varphi$.

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- Every orthoposet appears as some $\mathrm{Clop}(\Omega, \varphi)$ (Iturrioz 1982, Mayet 1982, Katrnoška 1982)

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Convex geometry: closure space (Ω, φ) such that ($\mathbf{x}$ closed, $p, q \in \Omega \setminus \mathbf{x}$, and $\varphi(\mathbf{x} \cup \{p\}) = \varphi(\mathbf{x} \cup \{q\})) \Rightarrow p = q$.

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Theorem (S. and W. 2012)

For (more general spaces than) finite convex geometries, the lattice $\mathrm{Reg}(\Omega, \varphi)$ is always pseudocomplemented.

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- Largely unexplored yet. Example of question: Does any Coxeter lattice of type D_n embed into some permutohedron P(N)?

Thanks for your attention !!!

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Luigi

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Luigi and