The extended permutohedron on a transitive relation

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- where we set

$$[n] \underset{\text{def.}}{=} \{1, 2, \ldots, n\}\,,$$

$$\mathcal{I}_n \underset{\text{def.}}{=} \{(i, j) \in [n] \times [n] \mid i < j\}\,,$$

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- Alternate definition: $\mathsf{P}(n) = \{\mathsf{Inv}(\sigma) \mid \sigma \in \mathfrak{S}_n\}$, ordered by $\subseteq$.

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- Conversely, every subset $\mathbf{x} \subseteq \mathcal{I}_n$, such that both $\mathbf{x}$ and $\mathcal{I}_n \setminus \mathbf{x}$ are transitive, is $\mathrm{Inv}(\sigma)$ for a unique $\sigma \in \mathfrak{S}_n$ (Dushnik and Miller 1941, Guilbaud and Rosenstiehl 1963).

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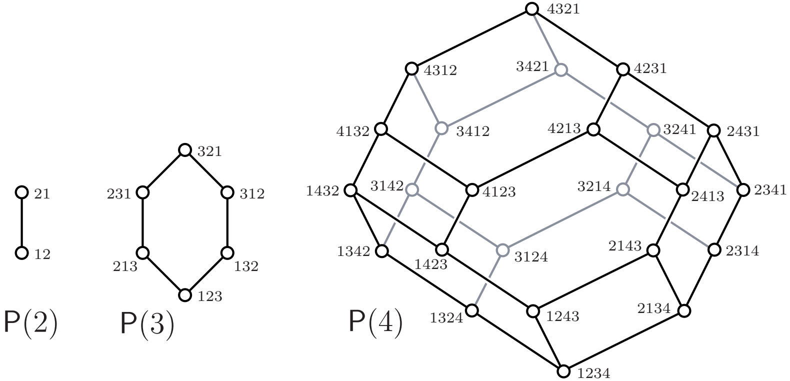
- Hence $\mathrm{P}(n) = \{\mathbf{x} \subseteq \mathcal{I}_n \mid \mathbf{x} \text{ is clopen}\}$, ordered by $\subseteq$.
- Observe that each $\mathbf{x} \in \mathrm{P}(n)$ is a strict ordering. It can be proved (Dushnik and Miller 1941) that those are exactly the finite strict orderings of order-dimension 2.

The permutohedra P(2), P(3), and P(4).



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Hence $\mathrm{P}(n)$ is an ortholattice.

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In particular, the join of $\{\mathbf{x}, \mathbf{y}\}$ in P(n) is $\mathrm{cl}(\mathbf{x} \cup \mathbf{y})$. Dually, the meet of $\{\mathbf{x}, \mathbf{y}\}$ in P(n) is $\mathrm{int}(\mathbf{x} \cap \mathbf{y})$.

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Semidistributivity means that
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The permutohedron $\mathrm{P}(n)$ is a bounded homomorphic image of a free lattice, for every positive integer n.

This means that there are a finitely generated free lattice F and a surjective lattice homomorphism $f\colon F \twoheadrightarrow \mathrm{P}(n)$ such that each $f^{-1}\{x\}$ has both a least and a largest element.

Basic definitions

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- Then we set

$$\mathrm{P}(E) \underset{\text{def.}}{=} \{\mathbf{a} \subseteq \boldsymbol{\delta}_E \mid \mathbf{a} \text{ is clopen}\}, \quad \text{(that's our guy)}$$

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- Obviously, $\mathrm{P}^*(E) \subseteq \mathrm{P}(E)$.
- Also, both $\mathrm{P}(E)$ and $\mathrm{P}^*(E)$ are orthocomplemented posets.

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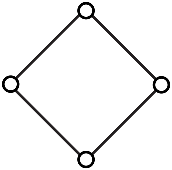
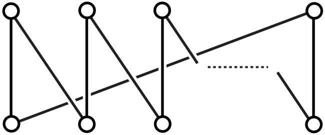
Illustrating square and crowns:





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- "Square-free" is just put there in order to ensure that $\mathrm{P}(E)$ be a lattice.
- For E an infinite chain, $\mathrm{P}(E)$ is not even semidistributive.

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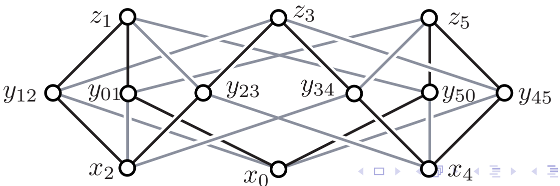

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Setting the problem

- Lattice-theoretical properties of $\mathrm{P}(E)$: make sense only in case $\mathrm{P}(E)$ is a lattice (duh), that is, E is square-free.
- Is there anything left in case E is not square-free?
- It turns out that yes.

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Operators cl and int defined as before: $\mathrm{cl}(\mathbf{x})$ is the transitive closure of $\mathbf{x}$, $\mathrm{int}(\mathbf{x}) = \mathbf{e} \setminus \mathrm{cl}(\mathbf{e} \setminus \mathbf{x})$.

The lattices $\mathrm{Reg}(\mathbf{e})$ and $\mathrm{Reg}_{\mathrm{op}}(\mathbf{e})$

Notation

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$\mathrm{Clop}(\mathbf{e})$ is an orthocomplemented poset. It may not be a lattice (e.g., $\mathrm{P}(E) = \mathrm{Clop}(\delta_E)$, for any poset E; take E non square-free).

Some notation

For a transitive relation **e** on a set E, write

$$x \lhd_{\mathbf{e}} y \underset{\text{def.}}{\Longleftrightarrow} (x, y) \in \mathbf{e}\,,$$
$$x \unlhd_{\mathbf{e}} y \underset{\text{def.}}{\Longleftrightarrow} (\text{either } x \lhd_{\mathbf{e}} y \text{ or } x = y)\,,$$

for all $x, y \in E$. We also set

$$[a, b]_{\mathbf{e}} = \{x \mid a \unlhd_{\mathbf{e}} x \text{ and } x \unlhd_{\mathbf{e}} b\}\,,$$
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for all $a, b \in E$. As $a \lhd_{\mathbf{e}} a$ may occur, a may belong to $]a, b]_{\mathbf{e}}$.

Square-free transitive relations

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A transitive relation $\mathbf{e}$ on a set E is square-free if the preordered set $(E, \trianglelefteq_{\mathbf{e}})$ is square-free. That is,

$$(\forall a, b, x, y)\Big(\big(a \trianglelefteq_{\mathbf{e}} x \text{ and } a \trianglelefteq_{\mathbf{e}} y \text{ and } x \trianglelefteq_{\mathbf{e}} b \text{ and } y \trianglelefteq_{\mathbf{e}} b\big) \Longrightarrow \big(\text{either } x \trianglelefteq_{\mathbf{e}} y \text{ or } y \trianglelefteq_{\mathbf{e}} x\big)\Big) .$$

When is Clop($\mathbf{e}$) a lattice?

Theorem (Santocanale and W 2012)

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- The particular case where **e** is full (i.e., $\mathbf{e} = E \times E$) follows from 2011 work by Hetyei and Krattenthaler. In that case, **e** is always square-free, and Clop(**e**) $=$ Reg(**e**) is denoted by Bip(E), the lattice of all bipartitions of a set E.

Permutohedra on non square-free posets

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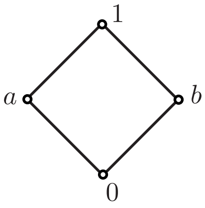
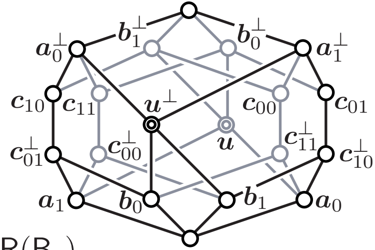
- In particular, $\mathrm{R}(E)$ is always a lattice.
- By earlier results, $\mathrm{P}(E)$ is a lattice, iff $\mathrm{P}(E) = \mathrm{R}(E)$, iff E is square-free.

The extended permutohedron on the square B_2

There it goes:



B_2



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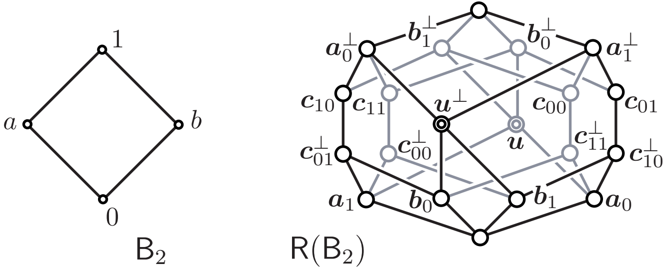

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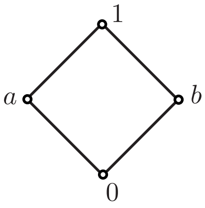

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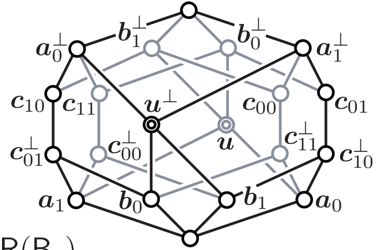

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- $\operatorname{card} R(B_2) = 20$ while $\operatorname{card} P(B_2) = 18$.
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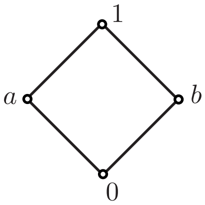

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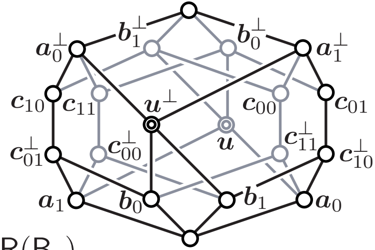

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- The two elements $\mathbf{u}$ and $\mathbf{u}^\perp$ of $R(B_2) \setminus P(B_2)$ are marked by doubled circles on the picture above.

Basic observations

- $\mathrm{Bip}(n) = \mathrm{Bip}([n])$ is the ortholattice of all binary relations $\mathbf{x}$ on $[n]$ that are both transitive and co-transitive, ordered by $\subseteq$.

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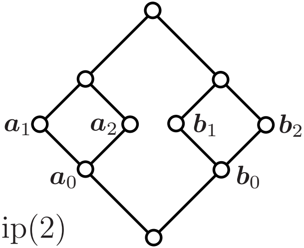
- Each $\mathrm{Bip}(n)$ is a graded lattice (Hetyei and Krattenthaler 2011).

Small bipartition lattices

- Here is a picture of Bip(2), together with the join-dependency relation on its join-irreducible elements.



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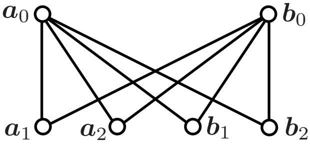

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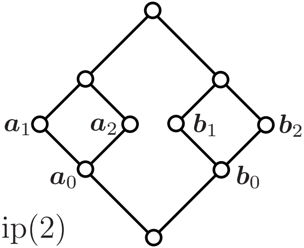

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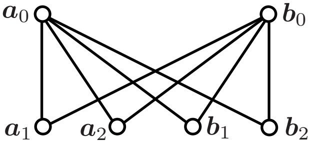

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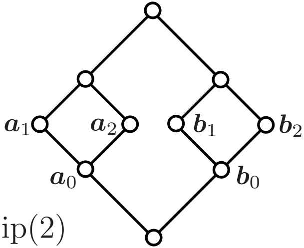

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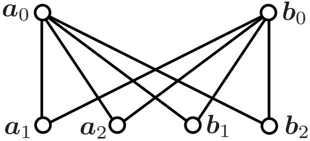

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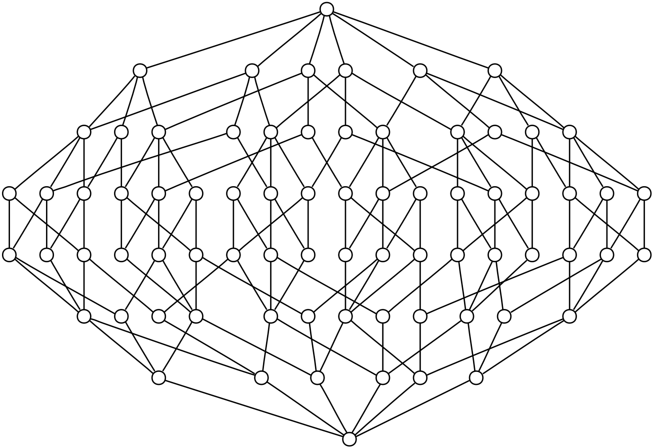
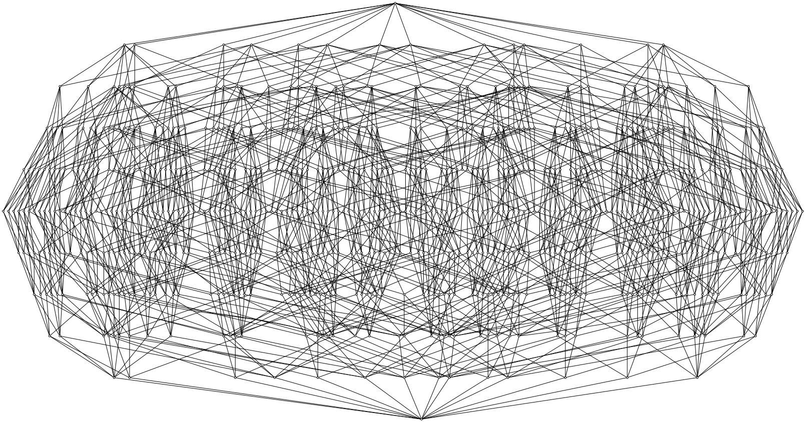
- In particular, Bip(2) is a bounded homomorphic image of a free lattice.
- This does not extend to higher bipartition lattices: for example, M_3 embeds into Bip(3), so Bip(3) is not even semidistributive.

The lattice Bip(3)



The lattice Bip(4)



Some open problems

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Is there a lattice (ortholattice) identity satisfied by every Bip(n)?

Some notation

- We denote by $\mathcal{C}(\mathbf{e})$ the set of all triples (a, b, U), where $(a, b) \in \mathbf{e}$, $U \subseteq [a, b]_{\mathbf{e}}$, and $a \neq b$ implies that $a \notin U$ and $b \in U$.

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for each $(a, b, U) \in \mathcal{C}(\mathbf{e})$.

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for each $(a, b, U) \in \mathcal{C}(\mathbf{e})$.
- Observe that $\langle a, b; U\rangle$ is bipartite (i.e., it cannot have both (x, y) and (y, z)) iff $a \neq b$. If $a = b$, say that $\langle a, b; U\rangle$ is a clepsydra.

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Reg($\mathbf{e}$) is the Dedekind-MacNeille completion of Clop($\mathbf{e}$), for any transitive relation $\mathbf{e}$.

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Lemma (Santocanale and W 2012)

Let $\mathbf{e}$ be a finite, antisymmetric, transitive relation and let $\mathbf{p}_i = \langle a_i, b_i; U_i \rangle$ be completely join-irreducible in Reg($\mathbf{e}$), for $i \in \{0, 1\}$. Then $\mathbf{p}_0 \, D \, \mathbf{p}_1$ in Reg($\mathbf{e}$) iff $[a_1, b_1]_{\mathbf{e}} \subsetneqq [a_0, b_0]_{\mathbf{e}}$ and $U_1 = ((U_0 \cap [a_1, b_1]_{\mathbf{e}}) \setminus \{a_1\}) \cup \{b_1\}$.

Corollary (Santocanale and W 2012)

The join-dependency relation on Reg($\mathbf{e}$) is a strict ordering, for any finite, antisymmetric, transitive relation $\mathbf{e}$.

Corollary (Santocanale and W 2012)

The lattice Reg($\mathbf{e}$) is a bounded homomorphic image of a free lattice, for any finite, antisymmetric, transitive relation $\mathbf{e}$.

Bounded lattices Reg($\mathbf{e}$)

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1. Reg($\mathbf{e}$) is a bounded homomorphic image of a free lattice.
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4. Every connected component of the preordering $\trianglelefteq_{\mathbf{e}}$ is either antisymmetric or has the form $\{a, b\}$ with $a \neq b$, $(a, b) \in \mathbf{e}$, and $(b, a) \in \mathbf{e}$.

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Hence if Reg(**e**) is a bounded homomorphic image of a free lattice, then it is a direct product of extended permutohedra on finite posets and copies of $\{0, 1\}$ and Bip(2).

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Is there a nontrivial lattice (ortholattice) identity that holds in $\mathrm{P}(n)$ for any positive integer n?

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- For $U \subseteq [n]$, denote by $\mathrm{A}_U(n)$ the set of all transitive $\mathbf{x} \in \mathcal{I}_n$ such that

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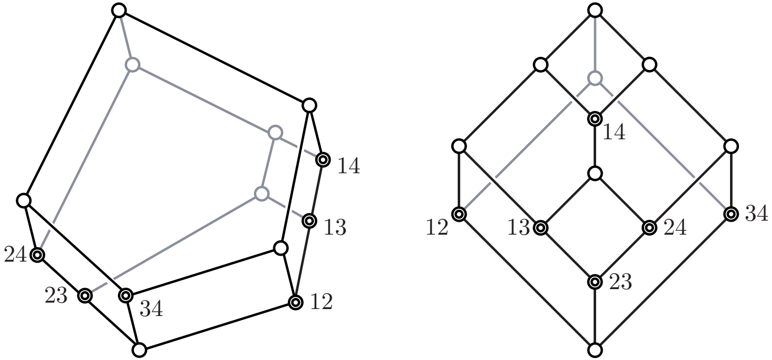
$\mathrm{A}_\varnothing(n) \cong \mathrm{A}_{[n]}(n)$ is the Tamari lattice on $n+1$ letters (associahedron).

Picturing the Cambrian lattices of type A, for $n = 4$



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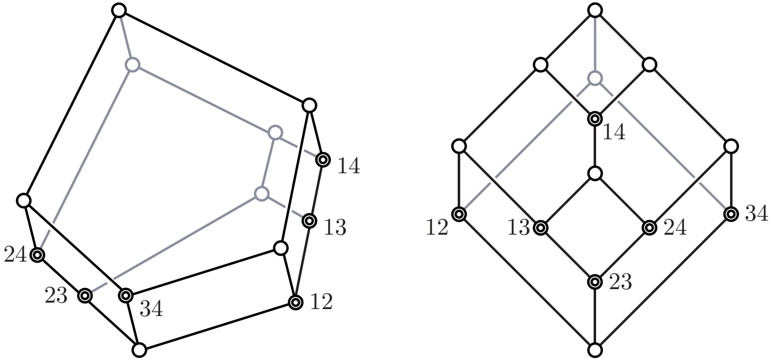

N. Reading observed that each $A_U(n)$ has cardinality $\frac{1}{n+1}\binom{2n}{n}$.

Minimal subdirect decomposition of Bip(n)

- $a \in [n]$ is isolated in $\mathbf{x} \in \mathrm{Bip}(n)$ if $((i, a) \in \mathbf{x}$ and $(a, i) \in \mathbf{x}) \Leftrightarrow i = a, \forall i \in [n]$.

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- For $0 \leq k < n$, $a \in [n]$, and $U \subseteq [n] \setminus \{a\}$ with k elements, denote $(\ldots)$ by $\mathrm{S}(n, k)$ the poset of all $\mathbf{x} \in \mathrm{Bip}(n)$ such that each isolated point of $\mathbf{x}$ is equal to a, and if a is isolated, then $(U^{c} \times \{a\}) \cup (\{a\} \times U) \subseteq \mathbf{x}$.

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Theorem (Santocanale and W 2012)

$\mathrm{Bip}(n)$ is a subdirect product of copies of the $\mathrm{S}(n,k)$ (minimal subdirect decomposition).

The bip-Cambrian lattices $\mathrm{S}(n, k)$

- Cardinalities for small values: $\operatorname{card} \mathrm{S}(3,0) = 24$, $\operatorname{card} \mathrm{S}(3,1) = 21$; $\operatorname{card} \mathrm{S}(4,0) = 158$, $\operatorname{card} \mathrm{S}(4,1) = 142$; $\operatorname{card} \mathrm{S}(5,0) = 1{,}320$, $\operatorname{card} \mathrm{S}(5,1) = 1{,}202$, $\operatorname{card} \mathrm{S}(5,2) = 1{,}198$.

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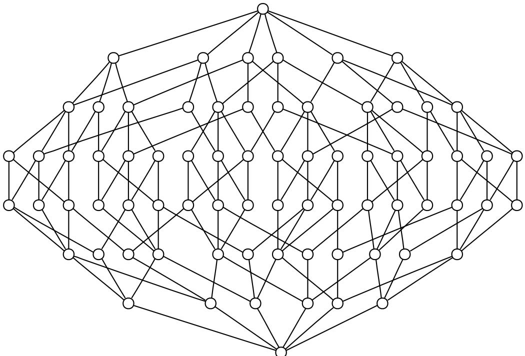
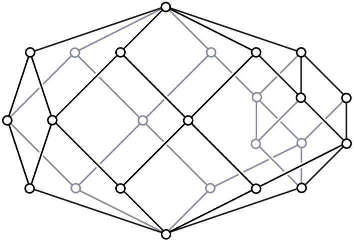
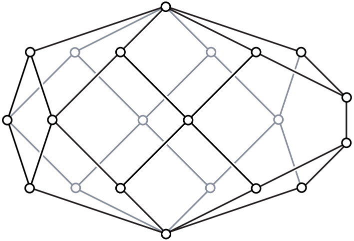
- Recall the picture of $\mathrm{Bip}(3)$:



Pictures of S(3, 0) and S(3, 1)





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Corollary (Santocanale and W 2012)

Let $n \geq 3$. Then the congruence lattice of Bip(n) is Boolean on $n \cdot 2^{n-1}$ atoms, with a top element added.