Von Neumann Coordinatization, Banaschewski functions, and larders

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Every geomodular lattice L is complemented, that is, for each $x \in L$, there exists $y \in L$ such that $x \vee y = 1$ (largest element of L) and $x \wedge y = 0$ (smallest element of L). (*Abbreviated* $x \oplus y = 1$, *and we say that* y *is a complement of* x.)

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A lattice is Arguesian, if it satisfies Desargues' identity. Every Arguesian lattice is modular, but the converse is false.

Illustrating Desargues' Rule

Desargues' Rule (on atoms of some modular lattices)

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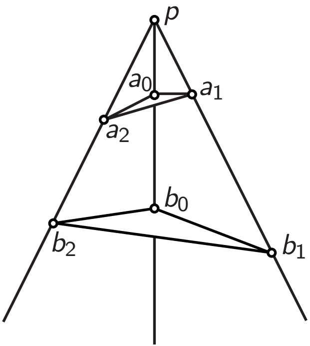

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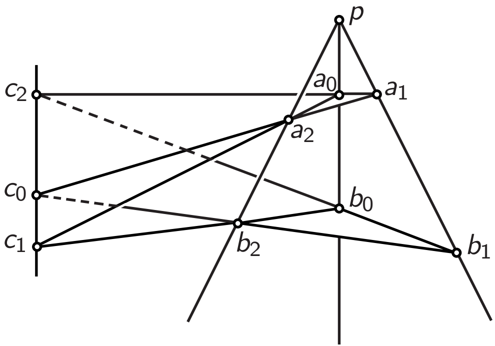

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- The normal subgroup lattice NSub G of any group G.
- The submodule lattice Sub M of any module M.
- (*more general*) Any lattice of permuting equivalence relations on a given set. (*Note: 'Arguesian' is then not the end of the story…*)

Fundamental examples of geomodular lattices

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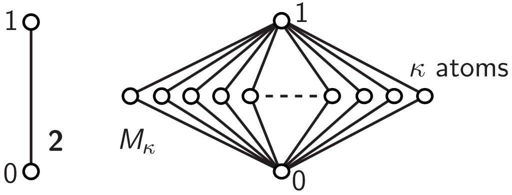

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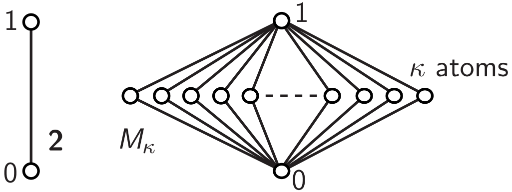

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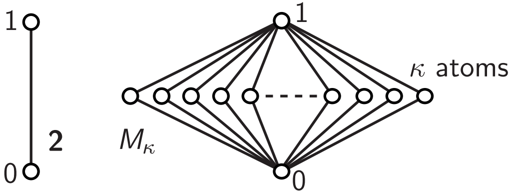

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(3) ... *and the non-Arguesian projective planes*!

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The decomposition above is unique.

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A bounded modular lattice is complemented iff it is sectionally complemented.

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Furthermore, $\mathbb{L}$ defines a functor from regular rings and ring homomorphisms to SCMLs and 0-lattice homomorphisms.

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So every coordinatizable lattice is sectionally complemented and modular. The easiest example of non-coordinatizable CML is M_7.

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A much more transparent proof of Jónsson's Coordinatization Theorem has recently been found by C. Herrmann.

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Von Neumann's condition requires the lattice have a unit, while Jónsson's does not. Nevertheless, Jónsson's Coordinatization Theorem is stated for lattices with unit.

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Banaschewski functions were first used in a simpler proof of Hahn's Embedding Theorem for totally ordered Abelian groups (embedding into a lexicographical power of the reals).

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Hence there exists a coordinatizable CML of cardinality $\aleph_1$ without a Banaschewski function.
However, in order to solve our coordinatization problem, more is needed...

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such that $y=x\oplus(y\ominus x)$ and $z\ominus x=(z\ominus y)\oplus(y\ominus x)$ for $x\leq y\leq z$ in X.

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That is, there exists no family $(I_{\alpha,\beta} \mid \alpha \leq \beta < \omega_1)$ of principal right ideals of $R_{\mathbb{F}}$ such that $I_{0,\alpha} = \tilde{\alpha} \cdot R_{\mathbb{F}}$ and $I_{\alpha,\gamma} = I_{\alpha,\beta} \oplus I_{\beta,\gamma}$ for all $\alpha \leq \beta \leq \gamma < \omega_1$.

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Each A_ξ is a CML with a large 4-frame (coming from the 1_4 matrix in e), thus (by Jónsson's Coordinatization Theorem) it is coordinatizable. However,

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But this can't be (use the result for $R_{\mathbb{F}}$ and $X_{\mathbb{F}}$).

Last remaining task

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Final blow:

Use CLL and larders (P. Gillibert and FW 2008).

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Informal statement of CLL

If $\Phi\vec{A}$ has no lifting wrt. Ψ $(= \mathbb{L})$, then construct (effectively!) a 'condensate' A of $\vec{A}$ such that $\Phi(A)$ has no lifting wrt. Ψ.

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(The latter requires a bit more care in the choice of the A_ξs.)

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- A SCML with a large 4-frame (or Arguesian with a large 3-frame) is coordinatizable iff it has a Banaschewski trace.

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