Large lower finite lattices with breadth three

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- We say that L is a k-ladder, if it is lower finite and every element of L has at most k lower covers.
- We say that L has breadth $\leq k$, if for every nonempty finite $X \subseteq L$, there exists $Y \subseteq X$ such that $|Y| \leq k$ and $\bigvee X = \bigvee Y$.

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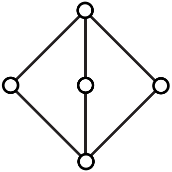
The lattice M_3 below has breadth 2. It is a 3-ladder but not a 2-ladder.



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Proposition (S. Z. Ditor, 1984)

Let k be a positive integer. Then every lower finite lattice L of breadth $\leq k$ has cardinality at most $\aleph_{k-1}$.

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- the congruence lattice of some lattice (A. P. Huhn 1989).
- the lattice of all normal subgroups of some locally finite group (P. Růžička, J. Tůma, and F. Wehrung 2006).

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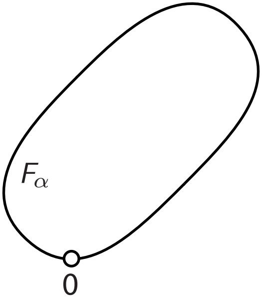

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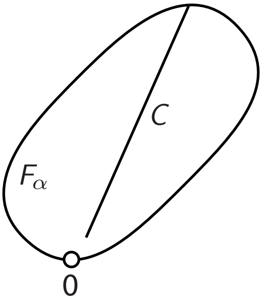

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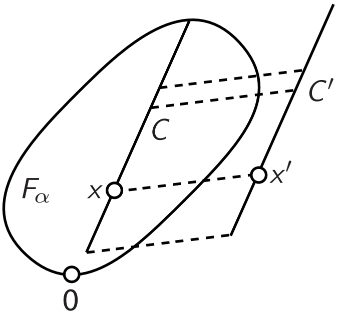

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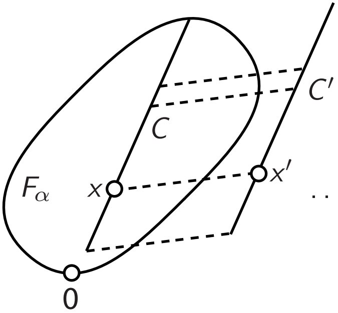

... and that's $F_{\alpha+1}$

Pick a cofinal chain C of F_α.
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And we are done ($F_{\alpha+1} := F_\alpha \cup C'$)!

Critical points: basic definitions

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- For classes $\mathcal{A}$ and $\mathcal{B}$ of algebras, denote by $\operatorname{crit}(\mathcal{A}, \mathcal{B})$ (critical point of $(\mathcal{A}, \mathcal{B})$) the least possible value of $|S|$ where $S \in \operatorname{Con_c} \mathcal{A} \setminus \operatorname{Con_c} \mathcal{B}$, if it exists; ∞, otherwise (i.e., if $\operatorname{Con_c} \mathcal{A} \subseteq \operatorname{Con_c} \mathcal{B}$).

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More specifically,

- $\mathrm{crit}(\mathcal{M}_3^{01}, \mathcal{D}^{01}) = \aleph_0$ and $\mathrm{crit}(\mathcal{M}_4^{01}, \mathcal{M}_3^{01}) = \aleph_2$ (M. Ploščica 2000, 2003) (later extended to unbounded lattices by P. Gillibert);

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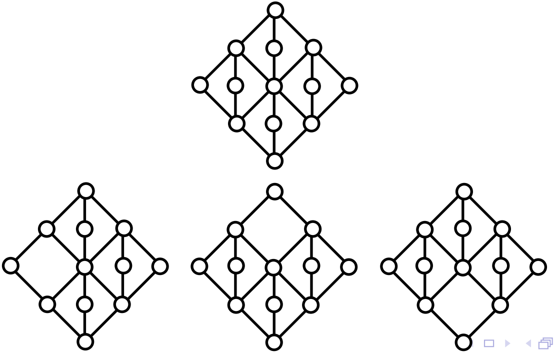

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Answer: nobody knows so far, but there's a feeling that 3-ladders of cardinality $\aleph_2$ could help.

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Partial answer (F. Wehrung 2008):

Yes, provided MA($\aleph_1$; precaliber $\aleph_1$) holds.

A first consistency result for 3-ladders of cardinality $\aleph_2$

Corollary (F. Wehrung 2008):

If MA($\aleph_1$; precaliber $\aleph_1$) holds, then there exists a 3-ladder of cardinality $\aleph_2$.

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• MA($\aleph_1$; precaliber $\aleph_1$) implies that $2^{\aleph_0} = 2^{\aleph_1}$ (Martin and Solovay, 1970). In particular, it contradicts the Continuum Hypothesis.

Simplified gap-1 morasses

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- $\beta - \alpha$:=unique ordinal ξ such that $\alpha + \xi = \beta$.
- For $\alpha \leq \beta$, define $\tau_{\alpha,\beta} \colon \beta \to \beta + (\beta - \alpha)$ by

$$\tau_{\alpha,\beta}(\xi) := \begin{cases} \xi\,, & \text{if } \xi < \alpha\,, \\ \beta + (\xi - \alpha)\,, & \text{if } \xi \geq \alpha\,. \end{cases}$$

Defining $(\kappa, 1)$-morasses

Definition (D. J. Velleman 1984)

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satisfying the following conditions:

(P0) (a) $\theta_0 = 2$, $0 < \theta_\alpha < \kappa$ for each $\alpha < \kappa$, and $\theta_\kappa = \kappa^+$.
(b) $\mathcal{F}_{\alpha,\beta}$ is a set of order-embeddings from θ_α into θ_β, for all $\alpha < \beta \leq \kappa$.

(P1) $|\mathcal{F}_{\alpha,\beta}| < \kappa$, for all $\alpha < \beta < \kappa$.

(P2) If $\alpha < \beta < \gamma \leq \kappa$, then
$\mathcal{F}_{\alpha,\gamma} = \{f \circ g \mid f \in \mathcal{F}_{\beta,\gamma} \text{ and } g \in \mathcal{F}_{\alpha,\beta}\}$.

(to be continued)

Defining simplified $(\kappa, 1)$-morasses (cont'd)

(end of definition of a simplified $(\kappa, 1)$-morass)

(P3) For each $\alpha < \kappa$, there exists a *nonzero* ordinal $\delta_\alpha < \theta_\alpha$ such that $\theta_{\alpha+1} = \theta_\alpha + (\theta_\alpha - \delta_\alpha)$ and $\mathcal{F}_{\alpha,\alpha+1} = \{\mathrm{id}_{\theta_\alpha}, \tau_{\delta_\alpha,\theta_\alpha}\}$.

(P4) For every limit ordinal $\lambda \leq \kappa$, all $\alpha_i < \lambda$ and $f_i \in \mathcal{F}_{\alpha_i,\lambda}$, for $i < 2$, there exists $\alpha < \lambda$ with $\alpha_0, \alpha_1 < \alpha$ together with $f_i' \in \mathcal{F}_{\alpha_i,\alpha}$, for $i < 2$, and $g \in \mathcal{F}_{\alpha,\lambda}$ such that $f_i = g \circ f_i'$ for each $i < 2$.

(P5) The equality $\theta_\alpha = \bigcup(f[\theta_\xi] \mid \xi < \alpha \text{ and } f \in \mathcal{F}_{\xi,\alpha})$ holds for each $\alpha > 0$.

Do these things exist at all?

Theorem

- Simplified $(\omega_1, 1)$-morasses exist in $\mathbf{L}[A]$, for each $A \subseteq \omega_1$ (R. Jensen 1970, K. Devlin 1984, and D. J. Velleman 1984).

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- If there exists no simplified $(\omega_1, 1)$-morass, then ω_2 is inaccessible in the constructible universe $\mathbf{L}$ (R. Jensen 1970, K. Devlin 1984, and D. J. Velleman 1984)
- If there exists an inaccessible cardinal, then there exists a generic extension without a "Kurepa tree", and thus without a simplified $(\omega_1, 1)$-morass (J. Silver 1971).

What does this have to do with 3-ladders?

Theorem (F. Wehrung 2008)

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The existence of a 3-ladder of cardinality $\aleph_2$ is consistent with both the Continuum Hypothesis and its negation.

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Eerie situation: The existence of a 3-ladder of cardinality $\aleph_2$ follows from either one of two axioms that are usually thought of as ‘orthogonal’ to each other.