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Lattice formulation
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The $\aleph_1$ case
A polarized metric on ℓ-representable lattices

The spectrum Problem for Abelian ℓ-groups: an overview

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ℓ-groups

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- In all what follows, restrict attention to Abelian ℓ-groups with order-unit (element $u \in G^+$ such that $(\forall x)(\exists n \in \mathbb{N})(x \leq nu)$).

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- $\operatorname{Spec}_\ell G \overset{\text{def}}{=} \{\text{prime } \ell\text{-ideals of } G\}$, topologized by the closed sets the $\{P \in \operatorname{Spec}_\ell G \mid X \subseteq P\}$ for $X \subseteq G$ (hull-kernel topology).

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Problem (Mundici 2011, but originating much earlier, e.g. Martínez 1973)

Describe the topological spaces of the form $\mathrm{Spec}_\ell\, G$, with G an Abelian ℓ-group with order-unit.

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- Spectral spaces are exactly the Zariski spectra of commutative unital rings (Hochster 1969)... not sufficient for ℓ-spectra!

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- Let us call such lattices ℓ-representable.

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ℓ-spectrum Problem (lattice-theoretical formulation)

Characterize ℓ-representable lattices

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  (Monteiro 1954).
- Every ℓ-representable lattice satisfies the following infinitary sentence (CBD, "countably based differences"):

$$(\forall a, b)(\exists_{n \in \mathbb{N}} c_n)(\forall x)\left(a \leq b \vee x \Leftrightarrow (\exists n \in \mathbb{N})(c_n \leq x)\right).$$

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- Delzell and Madden's 1994 counterexample is a completely normal lattice of cardinality $\aleph_1$, without CBD.

The countable case

The following is a full solution of the ℓ-spectrum Problem for countable lattices:

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Theorem (W 2019)

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- Fails for uncountable fields!

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- The class of all ℓ-representable lattices can be defined as

$$\ell\text{-}\mathbf{Rep} = \{D \mid (\exists f, G)(f \colon G^+ \to D \text{ induces an isomorphism } \mathrm{Id}_{\mathrm{c}}^{\ell} G \to D)\}\,.$$

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- Despite appearances, this is not a second-order existential characterization of ℓ-**Rep**: the condition "$f \colon G^+ \to D$ induces an isomorphism $\mathrm{Id}_c^\ell G \to D$" is $\mathscr{L}_{\omega_1\omega}$ (not $\mathscr{L}_{\omega\omega}$). In fact,

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Theorem (Di Nola and Lenzi 2020)

The class of all ℓ-representable lattices is not closed under ultrapowers. In particular, it is not the class of all models of a set of existential second-order sentences.

Projective vs. co-projective

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- A co-projective characterization would be of the form

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- By using the condensate construction, one then proves that this is not the case for ℓ-**Rep**. Thus: compl. normal + CBD not enough (starts at cardinality $\aleph_2$)!

Something always working at $\aleph_1$

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- Fails at cardinalities $\geq \aleph_2$.

A representation result in size $\aleph_1$

The following was considerably harder to get. It is a full solution of the ℓ-spectrum Problem for lattices of cardinality $\leq \aleph_1$:

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- The $\aleph_1$ work has no known extension to real spectra.

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- The class of all $\mathrm{Id}_{\mathrm{c}}^{\ell} G$, G Archimedean ℓ-group (not necessarily with unit), is not co-projective.

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- Not known for Archimedean ℓ-groups with unit.
- Reason for this: the arrows from the only known $\{0,1\}^3$-indexed non-commutative diagram of ℓ-groups, entailing, *via* condensates, the "non co-projective" statement, do not preserve order-units.

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Monotone-Cevian lattices

- A deviation $(x, y) \mapsto x \smallsetminus y$ (on a distributive lattice D) is monotone if it is order-preserving in x and order-reversing in y (e.g., $x_1 \leq x_2 \Rightarrow x_1 \smallsetminus y \leq x_2 \smallsetminus y$).

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There exists an Abelian (and even Archimedean) $\mathbb{Q}$-vector lattice G with order-unit, with $\aleph_1$ elements, such that $\mathrm{Id}_{\mathrm{c}}^{\ell}\, G$ has no monotone deviation.

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G has generators e_α $(0 \leq \alpha \leq \omega_1)$, with each $0 \leq e_\alpha \leq e_{\omega_1}$, and $0 < \gamma < \beta \Rightarrow e_\gamma \leq 2e_\beta$.