Projective classes as images of accessible functors

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References

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General type of problem

- We are given a class $\mathcal{C}$ of structures $\boldsymbol{M} = (M, \dots, R_1, R_2, R_3, \dots, f_1, f_2, f_3, \dots)$, in a given vocabulary $\mathbf{v} = (R_1, R_2, R_3, \dots, f_1, f_2, f_3, \dots)$ (the R_i are relation symbols, the f_i are operation symbols), usually "nicely defined".

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- We are asking whether there is a "nice" description for the class $\mathcal{C}\restriction_{\mathbf{u}}$ of all reducts $\boldsymbol{M}\restriction_{\mathbf{u}} = (M, \ldots, R_1, \ldots, f_1, \ldots)$ to a given subvocabulary $\mathbf{u} = (R_1, \ldots, f_1, \ldots)$, for $\boldsymbol{M} \in \mathcal{C}$.

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- Usually the class $\mathcal{C}$ is definable in (some extension of) first-order logic.
- More generally, given a "nice" class $\mathcal{C}$ of structures, we form a class $\mathcal{C}'$ by "forgetting some structure". Is $\mathcal{C}'$ "nice"?

Examples

1. $\mathcal{C}$ is the class of all totally ordered Abelian groups $(A, +, \leq)$ (i.e., $(A, +)$ is an Abelian group, $\leq$ is a total order on A, and $x \leq y \Rightarrow x + z \leq y + z$), $\mathbf{v} = (+, \leq)$.

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4. $\mathcal{C}$ is the class of all Abelian lattice-ordered groups ("ℓ-groups"). $\mathcal{C}'$ is the class of all lattices of principal ℓ-ideals of Abelian ℓ-groups (i.e., Stone duals of spectra of Abelian ℓ-groups).

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A closer look at Example (4)

- For an Abelian ℓ-group G, $\mathrm{Id_c}\, G \stackrel{\text{def}}{=}$ (lattice of all principal ℓ-ideals of G) $= \{\langle a\rangle \mid a \in G^+\}$ where $\langle a\rangle \stackrel{\text{def}}{=} \{x \in G \mid (\exists n \in \mathbb{N})(|x| \leq na)\}$. Let $\mathrm{Id_c}\,\mathcal{A} \stackrel{\text{def}}{=} \{D \mid (\exists G)(D \cong \mathrm{Id_c}\, G)\}$.

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- Every member of $\mathrm{Id_c}\, \mathcal{A}$ is a distributive 0-lattice. It is completely normal (abbrev. CN), that is, it satisfies

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- Every member of $\mathrm{Id_c}\, \mathcal{A}$ has countably based differences (abbrev. CBD), that is, it satisfies

$$(\forall a, b)(\exists_{n<\omega} c_n)(\forall x)(a \leq b \vee x \Leftrightarrow c_n \leq x \text{ for some } n)\,.$$

Ploščica's Condition

- For an ideal I in a distributive lattice D, $x \equiv_I y$ if $(\exists z \in I)(x \vee z = y \vee z)$. Set $D/I \stackrel{\text{def}}{=} D/{\equiv_I}$.

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- Question: Does the conjunction 0-DLat&CN&CBD&Plo (and more...) characterize the members of $\mathrm{Id_c}\,\mathcal{A}$?
- Answer: A strong NO under (a fragment of) GCH, with a counterexample of cardinality $\aleph_4$.

v-structures

- Vocabulary: $\mathbf{v} = (\mathbf{v}_{\mathrm{ope}}, \mathbf{v}_{\mathrm{rel}}, \mathsf{ar})$ with $\mathbf{v}_{\mathrm{ope}} \cap \mathbf{v}_{\mathrm{rel}} = \varnothing$ and $\mathsf{ar} \colon \mathbf{v}_{\mathrm{ope}} \cup \mathbf{v}_{\mathrm{rel}} \to$ ordinals (usually) with $0 \notin \mathsf{ar}[\mathbf{v}_{\mathrm{rel}}]$.

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- atomic formulas: $s = t$, for terms s and t, or $R(t_\xi \mid \xi \in \mathrm{ar}(R))$ where the t_ξ are terms and $R \in \mathbf{v}_{\mathrm{rel}}$.

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- $\mathscr{L}_{\kappa\lambda}$-elementary class:
  $\mathcal{C} = \mathbf{Mod}_{\mathbf{v}}(\mathrm{E}) \overset{\text{def}}{=} \{\boldsymbol{A} \in \mathbf{Str}(\mathbf{v}) \mid \boldsymbol{A} \models \mathrm{E}\}$ where E is an $\mathscr{L}_{\kappa\lambda}(\mathbf{v})$-sentence.

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- projective over $\mathscr{L}_{\kappa\lambda}$ (abbrev. $\mathrm{PC}(\mathscr{L}_{\kappa\lambda})$) if there are a vocabulary $\mathbf{w} \supseteq \mathbf{v}$ and a sentence $\mathsf{E} \in \mathscr{L}_{\kappa\lambda}(\mathbf{w})$ such that $\mathcal{C} = \{\boldsymbol{M}\restriction_{\mathbf{v}} \mid \boldsymbol{M} \in \mathbf{Mod}_{\mathbf{w}}(\mathsf{E})\}$.
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Theorem (W 2021)

Let λ be an infinite cardinal. Then $\mathrm{PC}(\mathscr{L}_{\infty\lambda}) = \mathrm{RPC}(\mathscr{L}_{\infty\lambda})$ (in full generality; no restrictions on vocabularies). Moreover, if λ is singular, then $\mathrm{PC}(\mathscr{L}_{\infty\lambda}) = \mathrm{PC}(\mathscr{L}_{\infty\lambda^+})$.

Examples of “elementary” classes

- Finiteness (of the ambiant universe) is $\mathscr{L}_{\omega_1\omega}$:

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- Torsion-freeness (of a group) is $\mathscr{L}_{\omega_1\omega}$:

$$\bigwedge_{0<n<\omega}(\forall x)(x^n=1\Rightarrow x=1)\,.$$

An example of RPC (that turns out to be PC)

- $\mathcal{C} \stackrel{\mathrm{def}}{=} \{\boldsymbol{M} = (M, \cdot, 1) \text{ monoid} \mid (\exists \boldsymbol{G} \text{ group})(\boldsymbol{M} \hookrightarrow \boldsymbol{G})\}$ is, by definition, $\mathrm{RPC}(\mathscr{L}_{\omega\omega})$.

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- Nonetheless, $\mathcal{C} = \{\boldsymbol{M} \mid (\exists \text{ group structure } \boldsymbol{G} \text{ on } M)(\exists f \colon \boldsymbol{M} \hookrightarrow \boldsymbol{G})\}$ is $\mathrm{PC}(\mathscr{L}_{\omega\omega})$.

Other examples

- For a unital ring R, $\operatorname{Id}_c R \stackrel{\text{def}}{=} (\vee, 0)$-semilattice of all finitely generated two-sided ideals of R. Let $\mathcal{C} \stackrel{\text{def}}{=} \{\operatorname{Id}_c R \mid R \text{ unital ring}\}$ (up to isomorphism).

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- Observe that they are all defined as images of functors.
- We will see that none of those classes is $\mathrm{co}\text{-}\mathrm{PC}(\mathscr{L}_{\infty\infty})$ (i.e., complement of a $\mathrm{PC}(\mathscr{L}_{\infty\infty})$).

Accessible categories and functors

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- There are many examples: $\mathbf{Str}(\mathbf{v})$, quasivarieties. . .

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- The assumption that **v** be λ-ary cannot be dispensed with (counterexamples for both directions, involving idempotence and emptiness, respectively).

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- Satisfaction of an $\mathscr{M}_{\kappa\lambda}(\mathbf{v})$-statement is expressed *via* the existence of a winning strategy in the associated game.

Tuuri's Interpolation Theorem

Theorem (Tuuri 1992)

Let κ be a regular cardinal, let **v** be a κ-ary vocabulary, set $\lambda \overset{\text{def}}{=} \sup\{\kappa^\alpha \mid \alpha < \kappa\}$, and let E and F be $\mathscr{L}_{\kappa^+\kappa}(\mathbf{v})$-sentences such that the conjunction E $\wedge$ F has no **v**-model. Then there exists an $\mathscr{M}_{\lambda^+\lambda}(\mathbf{v})$-sentence G, with vocabulary the intersection of the vocabularies of E and F, such that $\models$ (E $\Rightarrow$ G) and $\models$ (F $\Rightarrow$ $\sim$G).

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- By a 1971 counterexample due to Malitz, $\mathscr{M}_{\lambda^+\lambda}$ cannot be replaced by $\mathscr{L}_{\infty\infty}$ in the statement of Tuuri's Theorem.

Projective and co-projective

Corollary

Let **v** be a vocabulary. Then for all classes $\mathcal{A}$ and $\mathcal{B}$ of **v**-structures, if $\mathcal{A}$ is $\mathrm{PC}(\mathscr{L}_{\infty\infty})$, $\mathcal{B}$ is co-$\mathrm{PC}(\mathscr{L}_{\infty\infty})$, and $\mathcal{A} \subseteq \mathcal{B}$, then there exists an $\mathscr{M}_{\infty\infty}(\mathbf{v})$-sentence G such that $\mathcal{A} \subseteq \mathbf{Mod}_{\mathbf{v}}(\mathrm{G}) \subseteq \mathcal{B}$.

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In order to prove that a $\mathrm{PC}(\mathscr{L}_{\infty\infty})$ class $\mathcal{C}$ of **v**-structures is not $\text{co-PC}(\mathscr{L}_{\infty\infty})$, it suffices to prove that $\mathcal{C}$ is not $\mathscr{M}_{\infty\infty}(\mathbf{v})$-definable.

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But then, what is the advantage of $\mathscr{M}_{\infty\infty}$-definable over $\mathrm{PC}(\mathscr{L}_{\infty\infty})$-definable or $\text{co-}\mathrm{PC}(\mathscr{L}_{\infty\infty})$-definable?

That's back-and-forth!

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Definition (Karttunen 1979)

For a regular cardinal λ, a λ-back-and-forth system between models $\boldsymbol{M}$ and $\boldsymbol{N}$ over a vocabulary $\mathbf{v}$ consists of a poset $(\mathcal{F}, \trianglelefteq)$, together with a function $f \mapsto \overline{f}$ with domain $\mathcal{F}$, such that each $\overline{f} \colon \mathbf{d}(f) \overset{\cong}{\rightarrow} \mathbf{r}(f)$ with $\mathbf{d}(f) \leqslant \boldsymbol{M}$ and $\mathbf{r}(f) \leqslant \boldsymbol{N}$, and the following conditions hold:

1. $f \trianglelefteq g$ implies $\overline{f} \subseteq \overline{g}$;
2. $(\mathcal{F}, \trianglelefteq)$ is λ-inductive;
3. whenever $f \in \mathcal{F}$ and $x \in M$ (resp., $y \in N$), there is $g \in \mathcal{F}$ such that $f \subseteq g$ and $x \in \mathbf{d}(g)$ (resp., $y \in \mathbf{r}(g)$).

We then write $\boldsymbol{M} \leftrightarrows_{\lambda} \boldsymbol{N}$.

$\mathscr{M}_{\infty\lambda}$ versus back-and-forth

Theorem (Karttunen 1979)

Let λ be a regular cardinal and let $\boldsymbol{M}$ and $\boldsymbol{N}$ be structures over a vocabulary $\mathbf{v}$. If $\boldsymbol{M} \leftrightarrows_\lambda \boldsymbol{N}$, then $\boldsymbol{M}$ and $\boldsymbol{N}$ satisfy the same $\mathscr{M}_{\infty\lambda}(\mathbf{v})$-sentences.

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- Extended by Karttunen to the even more general languages $\mathscr{N}_{\infty\lambda}$.
- The syntax for $\mathscr{N}_{\infty\lambda}$ is far more complex than for $\mathscr{M}_{\infty\lambda}$, the semantics are even trickier (not unique!).

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- Applies to earlier introduced examples $\mathrm{Id_c}$(unital rings), $\mathrm{Id_c}$(Abelian ℓ-groups), duals of real spectra of commutative unital rings, and many others: each of those classes fails to be closed under a suitable $\leftrightarrows_\lambda$.

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- The real trouble is: find a back-and-forth system $\mathcal{F}\colon \boldsymbol{M} \leftrightarrows_\lambda \boldsymbol{N}$ with $\boldsymbol{M} \in \mathcal{C}$ and $\boldsymbol{N} \notin \mathcal{C}$ (where $\mathcal{C}$ is the given class).

Back-and-forth systems from continuous functors

- In many examples, such as Φ(unital rings) and Φ(Abelian ℓ-groups) (where $\Phi = \mathrm{Id}_{\mathrm{c}}$), $\leftrightarrows_\lambda$ arises from some λ-continuous functor $\Gamma\colon [\kappa]^{\mathrm{inj}} \to \mathcal{C}$ with $\kappa \geq \lambda$.

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- It is often the case that for $X \subseteq \kappa$ with $\operatorname{card} X < \lambda$, $\Gamma(X) = \Phi\left(\prod(S_{|u|} \mid u \in X^{\subseteq P})\right)$ (a "condensate"), where:
  1. P is a suitable finite lattice (in both examples above, $P = \{0,1\}^3$; also, this method provably fails for arbitrary finite bounded posets!);
  2. $X^{\subseteq P} \overset{\mathrm{def}}{=} \bigcup\{X^D \mid D \subseteq P\}$;
  3. $|u| \overset{\mathrm{def}}{=} \bigvee \operatorname{dom} u$ whenever $u \in X^{\subseteq P}$;
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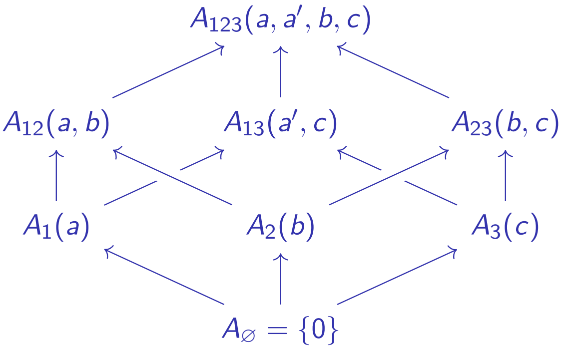
- Finding P and $\vec{S}$ is usually hard, very much connected to the algebraic and combinatorial data of the given problem.

The diagram $\vec{S}$ for $\mathrm{Id_c}$(Abelian ℓ-groups)



$0 \leq a \leq a' \leq 2a$; $b \geq 0$; $c \geq 0$.

$A_1(a) \to A_{13}(a', c)$ *via* $a \mapsto a'$.

Thanks for your attention!