Spectrum problems for structures arising from lattices and rings

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The spectrum of a commutative, unital ring

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- Is there an intrinsic characterization of the topological spaces of the form $\operatorname{Spec} A$?

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- Spec A is a spectral space, for every commutative unital ring A (well known and easy).

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- On the ring side, just consider unital ring homomorphisms.
- On the spectral space side, consider surjective spectral maps. For spectral spaces X and Y, a map $f\colon X \to Y$ is spectral if $f^{-1}[V] \in \mathring{\mathcal{K}}(X)$ whenever $V \in \mathring{\mathcal{K}}(Y)$.

Bounded distributive lattices

- A lattice is a structure $(L, \vee, \wedge)$, where $\vee$ and $\wedge$ are both binary operations on a set L such that there is a partial ordering $\leq$ for which $x \vee y = \sup(x, y)$ (the join of $\{x, y\}$) and $x \wedge y = \inf(x, y)$ (the meet of $\{x, y\}$) $\forall x, y \in L$.

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- A subset I in a bounded distributive lattice D is an ideal of D if $0 \in I$, $(\{x, y\} \subseteq I \Rightarrow x \vee y \in I)$, and $(\{x, y\} \cap I \neq \varnothing \Rightarrow x \wedge y \in I)$. An ideal I is prime if $I \neq D$ and $(x \wedge y \in I \Rightarrow \{x, y\} \cap I \neq \varnothing)$.

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- It is well known that the spectrum of any bounded distributive lattice is a spectral space.

The functors underlying Stone duality

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- For spectral spaces X and Y and a spectral map $\varphi\colon X \to Y$, the map $\mathring{\mathcal{K}}(\varphi)\colon \mathring{\mathcal{K}}(Y) \to \mathring{\mathcal{K}}(X)$, $V \mapsto \varphi^{-1}[V]$ is a $0,1$-lattice homomorphism.

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Theorem (Stone 1938)

The pair $(\operatorname{Spec}, \overset{\circ}{\mathcal{K}})$ induces a (categorical) duality, between bounded distributive lattices with $0,1$-lattice homomorphisms and spectral spaces with spectral maps.

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Note that in Hochster's Theorem's case, we do not obtain a duality (a ring is not determined by its spectrum).

Further spectra?

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- Further algebraic structures also afford a concept of spectrum.

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- An order-unit of G is an element $e \in G^+$ such that $G = \langle e \rangle$.

The ℓ-spectrum of an Abelian ℓ-group with unit

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- It turns out that more is true!

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Delzell and Madden's example is not second countable (i.e., no countable basis of the topology): in fact, it has $\operatorname{card} \mathring{\mathcal{K}}(X) = \aleph_1$.

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- Since G has an order-unit, $\operatorname{Id_c} G$ is a bounded distributive lattice.
- Thus we must prove that every countable completely normal bounded distributive lattice D is $\cong \operatorname{Id_c} G$ for some Abelian ℓ-group G with unit.

Very rough outline of the proof of the countable case (cont'd)

- The idea is to construct a "nice" surjective $0,1$-lattice homomorphism $f\colon \operatorname{Id_c} F_\omega \twoheadrightarrow D$, where F_ω denotes the free Abelian ℓ-group on a countably infinite generating set.

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For bounded distributive lattices A and B, a $0,1$-lattice homomorphism $f\colon A \to B$ is closed if whenever $a_0, a_1 \in A$ and $b \in B$, if $f(a_0) \leq f(a_1) \vee b$, then there exists $x \in A$ such that $a_0 \leq a_1 \vee x$ and $f(x) \leq b$.

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Very rough outline of the proof of the countable case (further cont'd)

- The map $f\colon \operatorname{Id_c} F_\omega \to D$ is constructed as $f = \bigcup_{n<\omega} f_n$ (each $f_n \subseteq f_{n+1}$), where each $f_n\colon L_n \to D$ is a lattice homomorphism, for a carefully constructed finite sublattice L_n of $\operatorname{Id_c} F_\omega$.

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- A crucial observation is that each $\mathrm{Op}(\mathcal{H})$ is a Heyting subalgebra of the Heyting algebra of all open subsets of $\mathbb{E}$.

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Analogous result for $\mathscr{L}_{\infty,\lambda}$ (for an infinite cardinal λ): proof currently under verification.

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- It turns out that $\operatorname{Spec}_{\mathrm{r}} A$ is a completely normal spectral space, for any commutative unital ring A.

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- Even the countable case of the problem above (i.e., for second countable spaces) is still open.

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Theorem (Mellor and Tressl 2012)

For any infinite cardinal λ, there is no $\mathscr{L}_{\infty,\lambda}$-characterization of the Stone duals of real spectra of commutative unital rings.

Subspaces of ℓ-spectra and real spectra

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Problem (W. 2017)

Is a retract of an ℓ-spectrum also an ℓ-spectrum? Same question for real spectra.

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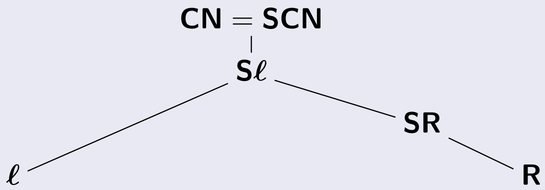
Theorem (W. 2017)

All containments and non-containments of the following picture are valid:



- All the separating counterexamples, intervening in the result above, have size $\aleph_1$, except for the counterexample constructed for $\mathbf{S}\ell \subsetneq \mathbf{CN}$, which has size $\aleph_2$.

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- Most of the examples constructed for the theorem above involve the construction of condensate (Gillibert and W. 2011), which turns diagram counterexamples to object counterexamples, with a jump of alephs corresponding to the order-dimension of the poset indexing the diagram (thus $\aleph_1$, $\aleph_2$, and so on).

Thanks for your attention!