Right-orderability versus left-orderability for monoids

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December 2020

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- The braid group B_3 is right- (and thus left-) orderable, but it is not bi-orderable (Dehornoy, Dynnikov, Rolfsen, and Wiest 2008).

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Proposition

- $X^{(1)}$ is positively right-orderable (*any total order works*).
- $X^{(1)}$ is bi-orderable iff it is left-orderable iff card $X \leq 2$.
- $X^{(1)}$ is positively bi-orderable iff it has a positive partial left order iff card $X \leq 1$.

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- Bi-orderability of an idempotent semigroup can be characterized by a finite list of forbidden subsemigoups (Saitô 1974).

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- If M is positively bi-orderable, then $ab = ba \in \{a, b\}$ for all idempotent $a, b \in M$. We then say that the idempotents of M form a chain.

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- Such structures (monoids with additional $x \mapsto x^\omega$) belong to L. Shevrin's epigroups (also often called completely π-regular semigroups).
- In any finite right-orderable monoid, $x^\omega = x^m$ where $x^m = x^{m+1}$ (the "period" of x is 1).

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- The latter (antisymmetry) can be extended to the case where any two idempotents commute (*much harder*).
- It fails in the general (finite) case.

Failure of antisymmetry in the finite, non-commutative case

$\cdot$	i	1	a	b	c	d	e	f	g	∞
i	i	i	a	b	c	d	e	f	g	∞
1	i	1	a	b	c	d	e	f	g	∞
a	a	a	a	b	d	d	e	f	g	∞
b	b	b	e	f	f	g	∞	∞	∞	∞
c	b	c	e	f	f	g	∞	∞	∞	∞
d	b	d	e	f	f	g	∞	∞	∞	∞
e	e	e	e	f	g	g	∞	∞	∞	∞
f	f	f	∞	∞	∞	∞	∞	∞	∞	∞
g	f	g	∞	∞	∞	∞	∞	∞	∞	∞
∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞

Table: A bi-orderable monoid, in which the idempotents form a chain, with no positive partial bi-order

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TFAE, for any cancellative commutative monoid M:

1. M is positively orderable;
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- An infinite, conical, orderable, commutative monoid may not have any positive partial order (W 2020).
- What about the finite commutative case?

Another finite counterexample

$+$	$\bar{1}$	0	1	2	3	4	5	6	∞
$\bar{1}$	$\bar{1}$	$\bar{1}$	1	2	2	4	5	5	∞
0	$\bar{1}$	0	1	2	3	4	5	6	∞
1	1	1	4	5	5	5	∞	∞	∞
2	2	2	5	5	5	∞	∞	∞	∞
3	2	3	5	5	6	∞	∞	∞	∞
4	4	4	5	∞	∞	∞	∞	∞	∞
5	5	5	∞	∞	∞	∞	∞	∞	∞
6	5	6	∞	∞	∞	∞	∞	∞	∞
∞	∞	∞	∞	∞	∞	∞	∞	∞	∞

Table: An orderable, but not positively orderable, commutative monoid (*with least possible cardinality*)

Two further finite examples (with "no best of two worlds")

$\cdot$	1	a	b	c
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Table: LO, positively RO, non bi-orderable, idempotent

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Table: Positively LO and RO, non bi-orderable

What about the cancellative case?

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- Holds trivially in the commutative case.
- General case: counterexample constructed in the following slides.

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- Play with non left-orderability for finite monoids. Isolate a "good reason" for non-orderability, which would not collide too "obviously" against cancellativity.

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- Such a "good reason" will take the form of a finite system of generators and relations, which will define a presentation of our monoid M.
- In order to prove right-orderability of M, express M as the universal monoid of a (cancellative) finite category, which will be, in some sense, right-orderable (order constructed directly).

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- Categories understood in the source/target (as opposed to domain/range) sense; $\partial_0 x$ =source of x, $\partial_1 x$ =target of x.

The finite category S generating M

So a category is a partial semigroup with "identity elements", subjected to certain rules (e.g., $xy\downarrow$ iff $\partial_1 x = \partial_0 y$; $x(yz)\downarrow$ iff $(xy)z\downarrow$ and then the two are equal; $\partial_0 x \cdot x = x \cdot \partial_1 x = x$; etc.).

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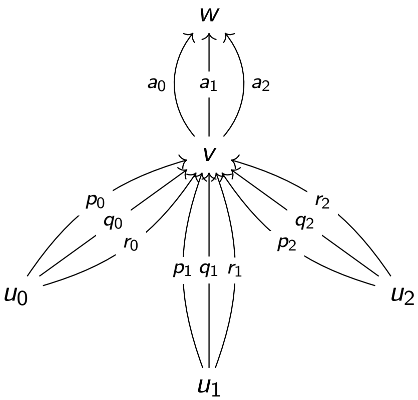

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- Universal monoid of S (denoted $\mathrm{U_{mon}}(S)$): universal with respect to all homomorphisms of S to a monoid (i.e., $xy\downarrow \Rightarrow f(xy) = f(x)f(y)$) sending all identities to 1.

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- $\mathrm{U}_{\mathrm{mon}}(S)$ consists of all finite sequences $x_0 x_1 \cdots x_n$, where all $x_i \in S$ and all $\partial_1 x_i \neq \partial_0 x_{i+1}$, with "contracted" concatenation;

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- Suppose that $\trianglelefteq$ is a left order on M. WMAT $a_0 \trianglelefteq a_1$ and $a_0 \trianglelefteq a_2$.

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- The picture above does not display the defining relations of S (e.g., $p_0 a_0 = r_0 a_2$, etc.).
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- $\mathrm{U}_{\mathrm{mon}}(S)$ consists of all finite sequences $x_0 x_1 \cdots x_n$, where all $x_i \in S$ and all $\partial_1 x_i \neq \partial_0 x_{i+1}$, with "contracted" concatenation; so $S \setminus \mathrm{Idt}\, S \hookrightarrow \mathrm{U}_{\mathrm{mon}}(S) \setminus \{1\}$.
- Suppose that $\trianglelefteq$ is a left order on M. WMAT $a_0 \trianglelefteq a_1$ and $a_0 \trianglelefteq a_2$. By left invariance,

$$p_0 a_0 \trianglelefteq p_0 a_1 = q_0 a_0 \trianglelefteq q_0 a_1 = r_0 a_0 \trianglelefteq r_0 a_2 = p_0 a_0 ,$$

so $p_0 a_0 = p_0 a_1$ in M, thus also in S, a contradiction.

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Group-embeddability criterion for $\mathrm{U}_{\mathrm{mon}}(S)$ (W 2018)

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$$p_i a_i = r_i a_{i+2}\,;\ p_i a_{i+1} = q_i a_i\,;\ q_i a_{i+1} = r_i a_i\,,\ \text{for } i \in \{0,1,2\}$$

(indices modulo 3).

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$$q_i = p_i a_{i+1} a_i^{-1}\,;\; r_i = p_i a_i a_{i+2}^{-1}\,;$$
$$a_{i+1} a_i^{-1} a_{i+1} = a_i a_{i+2}^{-1} a_i\,.$$

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- Hence $G \cong \mathrm{F_{gp}}(4) * (\mathbb{Z}/7\mathbb{Z})$, with $\mathrm{F_{gp}}(4)$ generated by $\{p_0, p_1, p_2, a_0\}$, $\mathbb{Z}/7\mathbb{Z}$ by c, $a_1 = a_0c$, $a_2 = a_0c^5$, and

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Right-orderability versus left-orderability for monoids

General

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The case of submonoids of groups

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- Therefore, M embeds into G.

Right-orderability of universal monoids

Lemma (W 2020)

Let S be a conical, right cancellative category, and let $\leq$ be a total order on $\overline{S}$:=canonical image of S in $\mathrm{U}_{\mathrm{mon}}(S)$, with least element 1, such that for all $x, y, z \in \overline{S}$, $x \leq y$ and $yz \in \overline{S}$ implies that $xz \in \overline{S}$ and $xz \leq yz$.

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$$1 < p_0 < q_0 < r_0 < p_1 < q_1 < r_1 < p_2 < q_2 < r_2 < a_0 < a_1 < a_2$$

(let this be the 0th chain).

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- In particular, M is positively right-orderable.

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Conclusion

- The monoid M embeds into a group, and its universal group is $\mathrm{F}_{\mathrm{gp}}(4) * (\mathbb{Z}/7\mathbb{Z})$ (*it has torsion*!).
- The monoid M is positively right-orderable.
- The monoid M is not left-orderable. In fact, there is no partial left order $\trianglelefteq$ of M for which $\{a_0, a_1, a_2\}$ has a least element.

Thanks for your attention!