Spectra of Abelian ℓ-groups are anti-elementary

Friedrich Wehrung

Université de Caen
LMNO, CNRS UMR 6139
Département de Mathématiques
14032 Caen cedex
E-mail: friedrich.wehrung01@unicaen.fr
URL: http://wehrungf.users.lmno.cnrs.fr

OAL-RAG 2019, May 2019

Working technique

Convex ℓ-subgroups and ℓ-ideals

- Our ℓ-groups will be denoted additively (even in the non-Abelian case).

Convex ℓ-subgroups and ℓ-ideals

- Our ℓ-groups will be denoted additively (even in the non-Abelian case).
- An ℓ-subgroup A in an ℓ-group G is *convex* if $\forall x \in A^+$ $[0, x] \subseteq A$.

Convex ℓ-subgroups and ℓ-ideals

- Our ℓ-groups will be denoted additively (even in the non-Abelian case).
- An ℓ-subgroup A in an ℓ-group G is *convex* if $\forall x \in A^+$ $[0, x] \subseteq A$.
- In addition, A is an ℓ-ideal if it is a normal subgroup.

Convex ℓ-subgroups and ℓ-ideals

- Our ℓ-groups will be denoted additively (even in the non-Abelian case).
- An ℓ-subgroup A in an ℓ-group G is *convex* if $\forall x \in A^+$ $[0, x] \subseteq A$.
- In addition, A is an ℓ-ideal if it is a normal subgroup.
- Denote by $\operatorname{Cs} G$ (resp., $\operatorname{Id} G$) the lattice of all convex ℓ-subgroups (resp., ℓ-ideals) of G.

Convex ℓ-subgroups and ℓ-ideals

- Our ℓ-groups will be denoted additively (even in the non-Abelian case).
- An ℓ-subgroup A in an ℓ-group G is *convex* if $\forall x \in A^+$ $[0, x] \subseteq A$.
- In addition, A is an ℓ-ideal if it is a normal subgroup.
- Denote by $\mathrm{Cs}\, G$ (resp., $\mathrm{Id}\, G$) the lattice of all convex ℓ-subgroups (resp., ℓ-ideals) of G.
- Then denote by $\mathrm{Cs_c}\, G$ (resp., $\mathrm{Id_c}\, G$) the (distributive) lattice (resp., $(\vee, 0)$-semilattice) of all finitely generated (equivalently, 1-generated) convex ℓ-subgroups (resp., ℓ-ideals) of G.

Convex ℓ-subgroups and ℓ-ideals

- Our ℓ-groups will be denoted additively (even in the non-Abelian case).
- An ℓ-subgroup A in an ℓ-group G is *convex* if $\forall x \in A^+$ $[0, x] \subseteq A$.
- In addition, A is an ℓ-ideal if it is a normal subgroup.
- Denote by $\operatorname{Cs} G$ (resp., $\operatorname{Id} G$) the lattice of all convex ℓ-subgroups (resp., ℓ-ideals) of G.
- Then denote by $\operatorname{Cs_c} G$ (resp., $\operatorname{Id_c} G$) the (distributive) lattice (resp., $(\vee, 0)$-semilattice) of all finitely generated (equivalently, 1-generated) convex ℓ-subgroups (resp., ℓ-ideals) of G.
- In particular, $\operatorname{Id} G \cong \operatorname{Con} G$ naturally.

Convex ℓ-subgroups and ℓ-ideals

- Our ℓ-groups will be denoted additively (even in the non-Abelian case).
- An ℓ-subgroup A in an ℓ-group G is *convex* if $\forall x \in A^+$ $[0, x] \subseteq A$.
- In addition, A is an ℓ-ideal if it is a normal subgroup.
- Denote by $\operatorname{Cs} G$ (resp., $\operatorname{Id} G$) the lattice of all convex ℓ-subgroups (resp., ℓ-ideals) of G.
- Then denote by $\operatorname{Cs_c} G$ (resp., $\operatorname{Id_c} G$) the (distributive) lattice (resp., $(\vee, 0)$-semilattice) of all finitely generated (equivalently, 1-generated) convex ℓ-subgroups (resp., ℓ-ideals) of G.
- In particular, $\operatorname{Id} G \cong \operatorname{Con} G$ naturally.
- If G is Abelian, then $\operatorname{Id_c} G$ is the Stone dual of the spectrum of G.

Convex ℓ-subgroups and ℓ-ideals

- Our ℓ-groups will be denoted additively (even in the non-Abelian case).
- An ℓ-subgroup A in an ℓ-group G is *convex* if $\forall x \in A^+$ $[0, x] \subseteq A$.
- In addition, A is an ℓ-ideal if it is a normal subgroup.
- Denote by $\operatorname{Cs} G$ (resp., $\operatorname{Id} G$) the lattice of all convex ℓ-subgroups (resp., ℓ-ideals) of G.
- Then denote by $\operatorname{Cs}_c G$ (resp., $\operatorname{Id}_c G$) the (distributive) lattice (resp., $(\vee, 0)$-semilattice) of all finitely generated (equivalently, 1-generated) convex ℓ-subgroups (resp., ℓ-ideals) of G.
- In particular, $\operatorname{Id} G \cong \operatorname{Con} G$ naturally.
- If G is Abelian, then $\operatorname{Id}_c G$ is the Stone dual of the spectrum of G.
- For $x \in G$, denote by $\langle x \rangle$ (resp., $\langle x \rangle^{\ell}$) the convex ℓ-subgroup (resp., ℓ-ideal) generated by x.

Complete normality

- A distributive lattice D with 0 is completely normal if $(\forall a, b \in D)\ (\exists x, y \in D)\ a \leq b \vee x,\ b \leq a \vee y$, and $x \wedge y = 0$.

Complete normality

- A distributive lattice D with 0 is completely normal if $(\forall a, b \in D)\ (\exists x, y \in D)\ a \leq b \vee x,\ b \leq a \vee y$, and $x \wedge y = 0$. This can also be stated by saying that the Stone dual of D is a root system (Monteiro 1954).

Complete normality

- A distributive lattice D with 0 is completely normal if $(\forall a, b \in D)\ (\exists x, y \in D)\ a \leq b \vee x,\ b \leq a \vee y$, and $x \wedge y = 0$. This can also be stated by saying that the Stone dual of D is a root system (Monteiro 1954).
- $\mathrm{Cs_c}\, G$ is completely normal, for every ℓ-group G.

Complete normality

- A distributive lattice D with 0 is completely normal if $(\forall a, b \in D)\ (\exists x, y \in D)\ a \leq b \vee x,\ b \leq a \vee y$, and $x \wedge y = 0$. This can also be stated by saying that the Stone dual of D is a root system (Monteiro 1954).
- $\mathrm{Cs_c}\, G$ is completely normal, for every ℓ-group G. (*Proof*: for $a, b \in G^+$, set $x \overset{\text{def}}{=} a \smallsetminus b = a - a \wedge b$, $y \overset{\text{def}}{=} b \smallsetminus a$. Then $\langle a \rangle \subseteq \langle b \rangle \vee \langle x \rangle$, $\langle b \rangle \subseteq \langle a \rangle \vee \langle y \rangle$, and $\langle x \rangle \cap \langle y \rangle = \{0\}$.)

Complete normality

- A distributive lattice D with 0 is completely normal if $(\forall a, b \in D)\ (\exists x, y \in D)\ a \leq b \vee x,\ b \leq a \vee y$, and $x \wedge y = 0$. This can also be stated by saying that the Stone dual of D is a root system (Monteiro 1954).
- $\mathrm{Cs_c}\, G$ is completely normal, for every ℓ-group G. (*Proof*: for $a, b \in G^+$, set $x \overset{\mathrm{def}}{=} a \smallsetminus b = a - a \wedge b$, $y \overset{\mathrm{def}}{=} b \smallsetminus a$. Then $\langle a \rangle \subseteq \langle b \rangle \vee \langle x \rangle$, $\langle b \rangle \subseteq \langle a \rangle \vee \langle y \rangle$, and $\langle x \rangle \cap \langle y \rangle = \{0\}$.)
- If G is representable (i.e., subdirect product of chains), then $\mathrm{Id_c}\, G$ is a homomorphic image of $\mathrm{Cs_c}\, G$ (via $\langle x \rangle \mapsto \langle x \rangle^{\ell}$), thus it is also completely normal.

Complete normality

- A distributive lattice D with 0 is completely normal if $(\forall a, b \in D)\ (\exists x, y \in D)\ a \leq b \vee x,\ b \leq a \vee y$, and $x \wedge y = 0$. This can also be stated by saying that the Stone dual of D is a root system (Monteiro 1954).
- $\mathrm{Cs_c}\, G$ is completely normal, for every ℓ-group G. (*Proof*: for $a, b \in G^+$, set $x \stackrel{\mathrm{def}}{=} a \smallsetminus b = a - a \wedge b$, $y \stackrel{\mathrm{def}}{=} b \smallsetminus a$. Then $\langle a \rangle \subseteq \langle b \rangle \vee \langle x \rangle$, $\langle b \rangle \subseteq \langle a \rangle \vee \langle y \rangle$, and $\langle x \rangle \cap \langle y \rangle = \{0\}$.)
- If G is representable (i.e., subdirect product of chains), then $\mathrm{Id_c}\, G$ is a homomorphic image of $\mathrm{Cs_c}\, G$ (via $\langle x \rangle \mapsto \langle x \rangle^{\ell}$), thus it is also completely normal.

Question addressed here:

Describe the lattices $\mathrm{Cs_c}\, G$ (resp., the $(\vee, 0)$-semilattices $\mathrm{Id_c}\, G$).

Some elements of positive answer

- An infinite lattice D is $\mathrm{Cs_c}\, G$ for some Abelian ℓ-group G iff there is an ℓ-group structure G on D and a surjective $f \colon G^+ \twoheadrightarrow D$ such that $f(x) \leq_D f(y)$ iff $x \in \langle y \rangle_G$.

Some elements of positive answer

- An infinite lattice D is $\mathrm{Cs_c}\, G$ for some Abelian ℓ-group G iff there is an ℓ-group structure G on D and a surjective $f\colon G^+ \twoheadrightarrow D$ such that $f(x) \leq_D f(y)$ iff $x \in \langle y \rangle_G$.
- Hence, the class of all lattices of the form $\mathrm{Cs_c}\, G$ has a second-order (also, projective class within $\mathscr{L}_{\omega_1\omega}$) characterization.

Some elements of positive answer

- An infinite lattice D is $\mathrm{Cs_c}\, G$ for some Abelian ℓ-group G iff there is an ℓ-group structure G on D and a surjective $f\colon G^+ \twoheadrightarrow D$ such that $f(x) \leq_D f(y)$ iff $x \in \langle y \rangle_G$.
- Hence, the class of all lattices of the form $\mathrm{Cs_c}\, G$ has a second-order (also, projective class within $\mathscr{L}_{\omega_1\omega}$) characterization.
- Every countable distributive $(\vee, 0)$-semilattice is $\mathrm{Id_c}\, G$ for some ℓ-group G (Růžička, Tůma, and W 2007).

Some elements of positive answer

- An infinite lattice D is $\mathrm{Cs_c}\, G$ for some Abelian ℓ-group G iff there is an ℓ-group structure G on D and a surjective $f\colon G^+ \twoheadrightarrow D$ such that $f(x) \leq_D f(y)$ iff $x \in \langle y \rangle_G$.
- Hence, the class of all lattices of the form $\mathrm{Cs_c}\, G$ has a second-order (also, projective class within $\mathscr{L}_{\omega_1\omega}$) characterization.
- Every countable distributive $(\vee, 0)$-semilattice is $\mathrm{Id_c}\, G$ for some ℓ-group G (Růžička, Tůma, and W 2007).
- A countable lattice is $\mathrm{Cs_c}\, G$, for some (Abelian) ℓ-group G, iff it is completely normal (W 2017).

Some elements of positive answer

- An infinite lattice D is $\mathrm{Cs_c}\, G$ for some Abelian ℓ-group G iff there is an ℓ-group structure G on D and a surjective $f\colon G^+ \twoheadrightarrow D$ such that $f(x) \leq_D f(y)$ iff $x \in \langle y \rangle_G$.
- Hence, the class of all lattices of the form $\mathrm{Cs_c}\, G$ has a second-order (also, projective class within $\mathscr{L}_{\omega_1 \omega}$) characterization.
- Every countable distributive $(\vee, 0)$-semilattice is $\mathrm{Id_c}\, G$ for some ℓ-group G (Růžička, Tůma, and W 2007).
- A countable lattice is $\mathrm{Cs_c}\, G$, for some (Abelian) ℓ-group G, iff it is completely normal (W 2017). G can be a vector lattice over any countable Archimedean totally ordered field $\mathbb{k}$; countability of $\mathbb{k}$ cannot be dispensed with.

Some elements of positive answer

- An infinite lattice D is $\mathrm{Cs_c}\, G$ for some Abelian ℓ-group G iff there is an ℓ-group structure G on D and a surjective $f\colon G^+ \twoheadrightarrow D$ such that $f(x) \leq_D f(y)$ iff $x \in \langle y \rangle_G$.
- Hence, the class of all lattices of the form $\mathrm{Cs_c}\, G$ has a second-order (also, projective class within $\mathscr{L}_{\omega_1\omega}$) characterization.
- Every countable distributive $(\vee, 0)$-semilattice is $\mathrm{Id_c}\, G$ for some ℓ-group G (Růžička, Tůma, and W 2007).
- A countable lattice is $\mathrm{Cs_c}\, G$, for some (Abelian) ℓ-group G, iff it is completely normal (W 2017). G can be a vector lattice over any countable Archimedean totally ordered field $\Bbbk$; countability of $\Bbbk$ cannot be dispensed with.
- The class of all lattices of the form $\mathrm{Cs_c}\, G$, for G Abelian, has no first-order characterization (W 2017).

Countably based differences

Definition

A distributive lattice D with 0 has countably based differences if $\forall a, b \in D$, $\exists (c_n)_{n<\omega} \in D^\omega$ such that $\forall x \in D$, $a \leq b \vee x$ iff $\exists n < \omega$ such that $c_n \leq x$.

Countably based differences

Definition

A distributive lattice D with 0 has countably based differences if $\forall a, b \in D$, $\exists (c_n)_{n<\omega} \in D^\omega$ such that $\forall x \in D$, $a \leq b \vee x$ iff $\exists n < \omega$ such that $c_n \leq x$.

- *This has been given a few other names in the literature, such as* $(\mathrm{Id}\,\omega)$ (*spectral spaces*) *and "*σ*-Conrad"* (*frames*).

Countably based differences

Definition

A distributive lattice D with 0 has countably based differences if $\forall a, b \in D$, $\exists (c_n)_{n<\omega} \in D^\omega$ such that $\forall x \in D$, $a \leq b \vee x$ iff $\exists n < \omega$ such that $c_n \leq x$.

- *This has been given a few other names in the literature, such as* $(\mathrm{Id}\,\omega)$ *(spectral spaces) and "σ-Conrad" (frames).*
- In particular, countably based differences can be expressed by an $\mathscr{L}_{\omega_1\omega_1}$ statement of lattice theory.

Countably based differences

Definition

A distributive lattice D with 0 has countably based differences if $\forall a, b \in D$, $\exists (c_n)_{n<\omega} \in D^\omega$ such that $\forall x \in D$, $a \leq b \vee x$ iff $\exists n < \omega$ such that $c_n \leq x$.

- *This has been given a few other names in the literature, such as* $(\mathrm{Id}\,\omega)$ (*spectral spaces*) *and "*σ*-Conrad"* (*frames*).
- In particular, countably based differences can be expressed by an $\mathscr{L}_{\omega_1\omega_1}$ statement of lattice theory.
- Every completely normal dual Heyting algebra is $\mathrm{Cs_c}\, G$ for some Abelian ℓ-group G (Cignoli, Gluschankof, and Lucas 2006, and Iberkleid, Martínez, and McGovern 2011).

Countably based differences

Definition

A distributive lattice D with 0 has countably based differences if $\forall a, b \in D$, $\exists (c_n)_{n<\omega} \in D^\omega$ such that $\forall x \in D$, $a \leq b \vee x$ iff $\exists n < \omega$ such that $c_n \leq x$.

- *This has been given a few other names in the literature, such as* $(\mathrm{Id}\,\omega)$ *(spectral spaces) and "σ-Conrad" (frames).*
- In particular, countably based differences can be expressed by an $\mathscr{L}_{\omega_1\omega_1}$ statement of lattice theory.
- Every completely normal dual Heyting algebra is $\mathrm{Cs_c}\, G$ for some Abelian ℓ-group G (Cignoli, Gluschankof, and Lucas 2006, and Iberkleid, Martínez, and McGovern 2011).
- For every Abelian ℓ-group G, $\mathrm{Id_c}\, G$ has countably based differences (refs above; *Proof*: given $a, b \in G^+$, set $\boldsymbol{c}_n \overset{\mathrm{def}}{=} \langle a \setminus nb \rangle \quad \forall n$).

Countably based differences

Definition

A distributive lattice D with 0 has countably based differences if $\forall a, b \in D$, $\exists (c_n)_{n<\omega} \in D^\omega$ such that $\forall x \in D$, $a \leq b \vee x$ iff $\exists n < \omega$ such that $c_n \leq x$.

- *This has been given a few other names in the literature, such as* ($\mathrm{Id}\,\omega$) (*spectral spaces*) *and "σ-Conrad"* (*frames*).
- In particular, countably based differences can be expressed by an $\mathscr{L}_{\omega_1\omega_1}$ statement of lattice theory.
- Every completely normal dual Heyting algebra is $\mathrm{Cs}_{\mathrm{c}}\, G$ for some Abelian ℓ-group G (Cignoli, Gluschankof, and Lucas 2006, and Iberkleid, Martínez, and McGovern 2011).
- For every Abelian ℓ-group G, $\mathrm{Id}_{\mathrm{c}}\, G$ has countably based differences (refs above; *Proof*: given $a, b \in G^+$, set $\boldsymbol{c}_n \stackrel{\mathrm{def}}{=} \langle a \smallsetminus nb \rangle \quad \forall n$). Question: how about the converse?

Ceva configurations

- We are given a totally ordered division ring $\mathbb{k}$.

Ceva configurations

- We are given a totally ordered division ring $\mathbb{k}$.
- Set $\mathbb{k}^+ \overset{\text{def}}{=} \{x \in \mathbb{k} \mid x \geq 0\}$, $\mathbb{k}^{++} \overset{\text{def}}{=} \{x \in \mathbb{k} \mid x > 0\}$, and $\overline{\mathbb{k}}^+ \overset{\text{def}}{=} \mathbb{k}^+ \cup \{+\infty\}$.

Ceva configurations

- We are given a totally ordered division ring $\mathbb{k}$.
- Set $\mathbb{k}^+ \overset{\text{def}}{=} \{x \in \mathbb{k} \mid x \geq 0\}$, $\mathbb{k}^{++} \overset{\text{def}}{=} \{x \in \mathbb{k} \mid x > 0\}$, and $\overline{\mathbb{k}}^+ \overset{\text{def}}{=} \mathbb{k}^+ \cup \{+\infty\}$.
- Denote by $\mathcal{O}(\overline{\mathbb{k}}^+)$ the (completely normal, distributive) lattice of all finite unions of intervals $[0, x[$, $]x, y[$, and $]y, +\infty]$ where $x, y \in \mathbb{k}^+$.

Ceva configurations

- We are given a totally ordered division ring $\mathbb{k}$.
- Set $\mathbb{k}^+ \stackrel{\text{def}}{=} \{x \in \mathbb{k} \mid x \geq 0\}$, $\mathbb{k}^{++} \stackrel{\text{def}}{=} \{x \in \mathbb{k} \mid x > 0\}$, and $\overline{\mathbb{k}}^+ \stackrel{\text{def}}{=} \mathbb{k}^+ \cup \{+\infty\}$.
- Denote by $\mathcal{O}(\overline{\mathbb{k}}^+)$ the (completely normal, distributive) lattice of all finite unions of intervals $[0, x[$, $]x, y[$, and $]y, +\infty]$ where $x, y \in \mathbb{k}^+$.
- Let us be given sets $U_{ij} \in \mathcal{O}(\overline{\mathbb{k}}^+)$, for $1 \leq i < j \leq 3$.

Ceva configurations

- We are given a totally ordered division ring $\mathbb{k}$.
- Set $\mathbb{k}^+ \overset{\text{def}}{=} \{x \in \mathbb{k} \mid x \geq 0\}$, $\mathbb{k}^{++} \overset{\text{def}}{=} \{x \in \mathbb{k} \mid x > 0\}$, and $\overline{\mathbb{k}}^+ \overset{\text{def}}{=} \mathbb{k}^+ \cup \{+\infty\}$.
- Denote by $\mathcal{O}(\overline{\mathbb{k}}^+)$ the (completely normal, distributive) lattice of all finite unions of intervals $[0, x[$, $]x, y[$, and $]y, +\infty]$ where $x, y \in \mathbb{k}^+$.
- Let us be given sets $U_{ij} \in \mathcal{O}(\overline{\mathbb{k}}^+)$, for $1 \leq i < j \leq 3$.
- For each of those, we set

$$C_{ij} \overset{\text{def}}{=} \{(x_1, x_2, x_3) \in (\mathbb{k}^+)^3 \mid (x_i, x_j) \neq (0,0) \text{ and } x_i^{-1} x_j \in U_{ij}\}$$

The sets U_{ij} and C_{ij}

Ceva configuration: when $C_{12} \cap C_{23} \subseteq C_{13} \subseteq C_{12} \cup C_{23}$.

The sets U_{ij} and C_{ij}

Ceva configuration: when $C_{12} \cap C_{23} \subseteq C_{13} \subseteq C_{12} \cup C_{23}$.

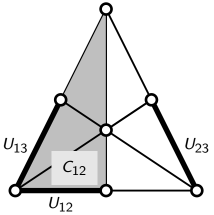


Figure: The sets C_{12} , C_{13} , and C_{23} in a Ceva configuration

The sets U_{ij} and C_{ij}

Ceva configuration: when $C_{12} \cap C_{23} \subseteq C_{13} \subseteq C_{12} \cup C_{23}$.

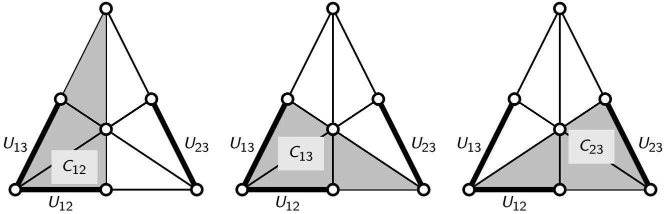


Figure: The sets C_{12} , C_{13} , and C_{23} in a Ceva configuration

This looks like the classical picture for Ceva's Theorem in affine geometry!

An (almost) lattice-theoretical Ceva

...and indeed:

An (almost) lattice-theoretical Ceva

…and indeed:

Proposition

Suppose that the following statements hold:

1. $0 \in U_{12} \cap U_{23} \cap U_{13}$;
2. $[0, \infty[\not\subseteq U_{12}$ and $[0, \infty[\not\subseteq U_{23}$;
3. $C_{12} \cap C_{23} \subseteq C_{13} \subseteq C_{12} \cup C_{23}$.

Then there are $x, y \in \Bbbk^{++}$ such that $U_{12} = [0, x[$, $U_{23} = [0, y[$, and $U_{13} = [0, xy[$.

In all the figures involved here, open polyhedral cones of $(\Bbbk^{+})^{3}$ are represented by their intersection with the 2-simplex

$$\{(x_1, x_2, x_3) \in (\Bbbk^{+})^{3} \mid x_1 + x_2 + x_3 = 1\},$$

and points are represented by their *homogeneous coordinates*.

In all the figures involved here, open polyhedral cones of $(\mathbb{k}^+)^3$ are represented by their intersection with the 2-simplex

$$\{(x_1, x_2, x_3) \in (\mathbb{k}^+)^3 \mid x_1 + x_2 + x_3 = 1\},$$

and points are represented by their *homogeneous coordinates*.

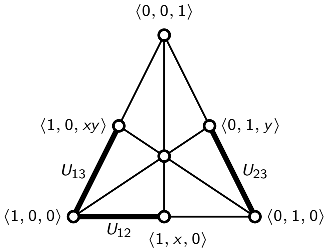


Figure: A Ceva configuration

Idea of proof

- Assume that $C_{12} \cap C_{23} \subseteq C_{13} \subseteq C_{12} \cup C_{23}$ (plus "boundary conditions").

Idea of proof

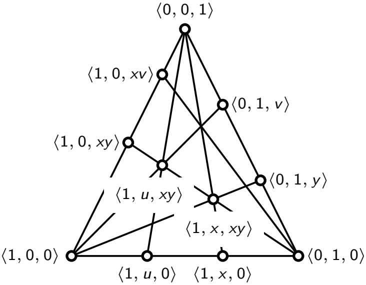
- Assume that $C_{12} \cap C_{23} \subseteq C_{13} \subseteq C_{12} \cup C_{23}$ (plus "boundary conditions").
- Eliminate the holes in U_{23} , then U_{12} , then U_{13} ,

Idea of proof

- Assume that $C_{12} \cap C_{23} \subseteq C_{13} \subseteq C_{12} \cup C_{23}$ (plus "boundary conditions").
- Eliminate the holes in U_{23}, then U_{12}, then U_{13}, in turn proving that they have the form $U_{23} = [0, y[$, $U_{12} = [0, x[$, then $U_{13} = [0, xy[$.

Idea of proof

- Assume that $C_{12} \cap C_{23} \subseteq C_{13} \subseteq C_{12} \cup C_{23}$ (plus "boundary conditions").
- Eliminate the holes in U_{23}, then U_{12}, then U_{13}, in turn proving that they have the form $U_{23} = [0, y[$, $U_{12} = [0, x[$, then $U_{13} = [0, xy[$.
- Typical picture intervening in the proof (here for U_{23}):



Introducing the diagram $\vec{A}$

- Introduce Abelian ℓ-groups A_p , for $p \in \mathfrak{P}[3] \stackrel{\text{def}}{=} \{\varnothing, 1, 2, 3, 12, 13, 23, 123\}$.

Introducing the diagram $\vec{A}$

- Introduce Abelian ℓ-groups A_p , for $p \in \mathfrak{P}[3] \stackrel{\mathrm{def}}{=} \{\varnothing, 1, 2, 3, 12, 13, 23, 123\}$.
- In the figure below, each A_p is written with its canonical generating subset (e.g., A_{123} is generated by $\{a, a', b, c\}$).

Introducing the diagram $\vec{A}$

- Introduce Abelian ℓ-groups A_p , for $p \in \mathfrak{P}[3] \stackrel{\text{def}}{=} \{\varnothing, 1, 2, 3, 12, 13, 23, 123\}$.
- In the figure below, each A_p is written with its canonical generating subset (e.g., A_{123} is generated by $\{a, a', b, c\}$).
- The relations, defining those ℓ-groups, are $0 \leq a \leq a' \leq 2a$, $0 \leq b$, and $0 \leq c$.

Introducing the diagram $\vec{A}$

- Introduce Abelian ℓ-groups A_p , for $p \in \mathfrak{P}[3] \overset{\mathrm{def}}{=} \{\varnothing, 1, 2, 3, 12, 13, 23, 123\}$.
- In the figure below, each A_p is written with its canonical generating subset (e.g., A_{123} is generated by $\{a, a', b, c\}$).
- The relations, defining those ℓ-groups, are $0 \leq a \leq a' \leq 2a$, $0 \leq b$, and $0 \leq c$.

$$
\begin{tikzcd}
 & A_{123}(a, a', b, c) & \\
A_{12}(a, b) \arrow[ur, "\alpha_{12}^{123}"] & A_{13}(a', c) \arrow[u, "\alpha_{13}^{123}"] & A_{23}(b, c) \arrow[ul, "\alpha_{23}^{123}"'] \\
A_1(a) \arrow[u, "\alpha_1^{12}"] \arrow[ur, "\boldsymbol{\alpha_1^{13}}"] & A_2(b) \arrow[ul, "\alpha_2^{12}"'] \arrow[ur, "\alpha_2^{23}"] & A_3(c) \arrow[ul, "\alpha_3^{13}"'] \arrow[u, "\alpha_3^{23}"'] \\
 & A_{\varnothing} = \{0\} \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}
$$

$\alpha_1^{13}(a) \overset{\mathrm{def}}{=} a'$

Each $\alpha_{\varnothing}^{p}$ is the zero map $A_{\varnothing} \to A_p$

The diagram $\vec{A}$ (cont'd)

- In particular, $\vec{A}(1, 123)$ consists of two ℓ-embeddings (resp., $a \mapsto a$ and $a \mapsto a'$).

The diagram $\vec{A}$ (cont'd)

- In particular, $\vec{A}(1, 123)$ consists of two ℓ-embeddings (resp., $a \mapsto a$ and $a \mapsto a'$).
- In that sense the diagram $\vec{A}$ is non-commutative.

The diagram $\vec{A}$ (cont'd)

- In particular, $\vec{A}(1, 123)$ consists of two ℓ-embeddings (resp., $a \mapsto a$ and $a \mapsto a'$).
- In that sense the diagram $\vec{A}$ is non-commutative. Its arrows do not preserve units.

The diagram $\vec{A}$ (cont'd)

- In particular, $\vec{A}(1, 123)$ consists of two ℓ-embeddings (resp., $a \mapsto a$ and $a \mapsto a'$).
- In that sense the diagram $\vec{A}$ is non-commutative. Its arrows do not preserve units.

Proposition

The diagram $\vec{A}$ is Id_c-commutative, in the sense that for every set I, the diagram $\mathrm{Id}_c\,\vec{A}^I$ (based on the poset $\mathfrak{P}[3]^I$) is commutative.

The diagram $\vec{A}$ (cont'd)

- In particular, $\vec{A}(1, 123)$ consists of two ℓ-embeddings (resp., $a \mapsto a$ and $a \mapsto a'$).
- In that sense the diagram $\vec{A}$ is non-commutative. Its arrows do not preserve units.

Proposition

The diagram $\vec{A}$ is Id_{c}-commutative, in the sense that for every set I, the diagram $\mathrm{Id}_{\mathrm{c}}\,\vec{A}^I$ (based on the poset $\mathfrak{P}[3]^I$) is commutative. *Accordingly, we set* $\vec{\boldsymbol{A}} \overset{\text{def}}{=} \mathrm{Id}_{\mathrm{c}}\,\vec{A}$.

The diagram $\vec{A}$ (cont'd)

- In particular, $\vec{A}(1, 123)$ consists of two ℓ-embeddings (resp., $a \mapsto a$ and $a \mapsto a'$).
- In that sense the diagram $\vec{A}$ is non-commutative. Its arrows do not preserve units.

Proposition

The diagram $\vec{A}$ is Id_c-commutative, in the sense that for every set I, the diagram $\mathrm{Id}_c\,\vec{A}^I$ (based on the poset $\mathfrak{P}[3]^I$) is commutative. *Accordingly, we set* $\vec{\boldsymbol{A}} \overset{\mathrm{def}}{=} \mathrm{Id}_c\,\vec{A}$.

Main idea of the proof.

From $0 \leq a \leq a' \leq 2a$ it follows that $\alpha_{12}^{123} \circ \alpha_1^{12} \leq \alpha_{13}^{123} \circ \alpha_1^{13} \leq 2 \cdot (\alpha_{12}^{123} \circ \alpha_1^{12})$. Thus, for all arrows f and g between two nodes in $\mathfrak{P}[3]^I$, $f \leq 2g$ and $g \leq 2f$. $\square$

The main negative property of $\vec{A}$

For $p \leq q$ in $\mathfrak{P}[3]$, we shall denote by $\boldsymbol{\alpha}_p^q$ the unique arrow from $\boldsymbol{A}_p$ to $\boldsymbol{A}_q$ in $\vec{\boldsymbol{A}}$. For example, $\boldsymbol{\alpha}_1^{123} = \mathrm{Id}_{\mathrm{c}}(\alpha_{12}^{123} \circ \alpha_1^{12}) = \mathrm{Id}_{\mathrm{c}}(\alpha_{13}^{123} \circ \alpha_1^{13})$.

The main negative property of $\vec{A}$

For $p \leq q$ in $\mathfrak{P}[3]$, we shall denote by $\boldsymbol{\alpha}_p^q$ the unique arrow from $\boldsymbol{A}_p$ to $\boldsymbol{A}_q$ in $\vec{\boldsymbol{A}}$. For example, $\boldsymbol{\alpha}_1^{123} = \mathrm{Id}_{\mathrm{c}}(\alpha_{12}^{123} \circ \alpha_1^{12}) = \mathrm{Id}_{\mathrm{c}}(\alpha_{13}^{123} \circ \alpha_1^{13})$. The elements

$$\boldsymbol{a}_1 \overset{\mathrm{def}}{=} \langle a \rangle_{A_{123}} = \langle a' \rangle_{A_{123}}\,, \quad \boldsymbol{a}_2 \overset{\mathrm{def}}{=} \langle b \rangle_{A_{123}}\,, \quad \boldsymbol{a}_3 \overset{\mathrm{def}}{=} \langle c \rangle_{A_{123}}$$

all belong to $\boldsymbol{A}_{123}$.

The main negative property of $\vec{A}$

For $p \leq q$ in $\mathfrak{P}[3]$, we shall denote by $\boldsymbol{\alpha}_p^q$ the unique arrow from $\boldsymbol{A}_p$ to $\boldsymbol{A}_q$ in $\vec{\boldsymbol{A}}$. For example, $\boldsymbol{\alpha}_1^{123} = \mathrm{Id}_{\mathrm{c}}(\alpha_{12}^{123} \circ \alpha_1^{12}) = \mathrm{Id}_{\mathrm{c}}(\alpha_{13}^{123} \circ \alpha_1^{13})$. The elements

$$\boldsymbol{a}_1 \overset{\text{def}}{=} \langle a \rangle_{A_{123}} = \langle a' \rangle_{A_{123}}\,, \quad \boldsymbol{a}_2 \overset{\text{def}}{=} \langle b \rangle_{A_{123}}\,, \quad \boldsymbol{a}_3 \overset{\text{def}}{=} \langle c \rangle_{A_{123}}$$

all belong to $\boldsymbol{A}_{123}$.

Lemma

There is no family $(\boldsymbol{c}_{ij} \mid i \neq j \text{ in } \{1,2,3\})$ of elements of $\boldsymbol{A}_{123}$ satisfying the following statements:

1. Each $\boldsymbol{c}_{ij}$ belongs to the range of $\boldsymbol{\alpha}_{ij}^{123}$.
2. $\boldsymbol{a}_i \leq \boldsymbol{a}_j \vee \boldsymbol{c}_{ij}$ whenever $\{i,j\}$ is either $\{1,2\}$ or $\{2,3\}$.
3. $\boldsymbol{c}_{ij} \wedge \boldsymbol{c}_{ji} = 0$ whenever $\{i,j\}$ is either $\{1,2\}$ or $\{2,3\}$.
4. $\boldsymbol{c}_{12} \wedge \boldsymbol{c}_{23} \leq \boldsymbol{c}_{13} \leq \boldsymbol{c}_{12} \vee \boldsymbol{c}_{23}$.

Idea of proof

- Represent the elements of each $\boldsymbol{A}_p \overset{\mathrm{def}}{=} \mathrm{Id}_c\, A_p$, for $p \in \mathfrak{P}[3]$, by (relatively) open cones, using Baker-Beynon duality.

Idea of proof

- Represent the elements of each $\boldsymbol{A}_p \stackrel{\mathrm{def}}{=} \mathrm{Id_c}\, A_p$, for $p \in \mathfrak{P}[3]$, by (relatively) open cones, using Baker-Beynon duality.
- Don't touch $\boldsymbol{A}_p$ for $p \neq 123$. Now for $p = 123$, we collapse $\boldsymbol{A}_{123}$ by identifying a and a'.

Idea of proof

- Represent the elements of each $\boldsymbol{A}_p \overset{\text{def}}{=} \mathrm{Id_c}\, A_p$, for $p \in \mathfrak{P}[3]$, by (relatively) open cones, using Baker-Beynon duality.
- Don't touch $\boldsymbol{A}_p$ for $p \neq 123$. Now for $p = 123$, we collapse $\boldsymbol{A}_{123}$ by identifying a and a'.
- This way, we send $\boldsymbol{A}_\varnothing$ to $\{0\}$, $\boldsymbol{A}_i$ to $\{0, 1\}$ for $i \in \{1, 2, 3\}$, and all other $\boldsymbol{A}_p$ to $\mathcal{O}_k \overset{\text{def}}{=}$ lattice of all rational strict open polyhedral cones of $(\mathbb{Q}^+)^k$, with k the cardinality of p (either 2 or 3).

Idea of proof

- Represent the elements of each $\boldsymbol{A}_p \overset{\text{def}}{=} \mathrm{Id}_c\, A_p$, for $p \in \mathfrak{P}[3]$, by (relatively) open cones, using Baker-Beynon duality.
- Don't touch $\boldsymbol{A}_p$ for $p \neq 123$. Now for $p = 123$, we collapse $\boldsymbol{A}_{123}$ by identifying a and a'.
- This way, we send $\boldsymbol{A}_\varnothing$ to $\{0\}$, $\boldsymbol{A}_i$ to $\{0, 1\}$ for $i \in \{1, 2, 3\}$, and all other $\boldsymbol{A}_p$ to $\mathcal{O}_k \overset{\text{def}}{=}$ lattice of all rational strict open polyhedral cones of $(\mathbb{Q}^+)^k$, with k the cardinality of p (either 2 or 3).
- This way, such inequalities as $\boldsymbol{c}_{12} \wedge \boldsymbol{c}_{23} \leq \boldsymbol{c}_{13} \leq \boldsymbol{c}_{12} \vee \boldsymbol{c}_{23}$ translate to conditions like $C_{12} \cap C_{23} \subseteq C_{13} \subseteq C_{12} \cup C_{23}$ (C_{ij} arising from $\boldsymbol{c}_{ij}$).

Idea of proof (cont'd)

- The condition $\boldsymbol{c}_{ij} \in \operatorname{rng} \boldsymbol{\alpha}_{ij}^{123}$ translates to

  $$C_{ij} = \{(x_1, x_2, x_3) \in (\mathbb{Q}^+)^3 \mid (x_i, x_j) \neq (0,0) \text{ and } x_i^{-1} x_j \in U_{ij}\}$$

  for suitable U_{ij} .

Idea of proof (cont'd)

- The condition $\boldsymbol{c}_{ij} \in \operatorname{rng} \boldsymbol{\alpha}_{ij}^{123}$ translates to

  $$C_{ij} = \{(x_1, x_2, x_3) \in (\mathbb{Q}^+)^3 \mid (x_i, x_j) \neq (0,0) \text{ and } x_i^{-1} x_j \in U_{ij}\}$$

  for suitable U_{ij} .
- This way, the given conditions translate to the geometric conditions stated in "lattice Ceva".

Idea of proof (cont'd)

- The condition $\boldsymbol{c}_{ij} \in \operatorname{rng} \boldsymbol{\alpha}_{ij}^{123}$ translates to

  $$C_{ij} = \{(x_1, x_2, x_3) \in (\mathbb{Q}^+)^3 \mid (x_i, x_j) \neq (0, 0) \text{ and } x_i^{-1} x_j \in U_{ij}\}$$

  for suitable U_{ij} .
- This way, the given conditions translate to the geometric conditions stated in "lattice Ceva".
- By the latter, we are lead to $\lambda, \mu \in \mathbb{Q}^{++}$ such that (up to identifications) $\boldsymbol{c}_{12} = \langle \lambda a \smallsetminus b \rangle$, $\boldsymbol{c}_{23} = \langle \mu b \smallsetminus c \rangle$, and $\boldsymbol{c}_{13} = \langle \lambda \mu a' \smallsetminus c \rangle$ (*recall that* $x \smallsetminus y = x - x \wedge y$).

Idea of proof (cont'd)

- The condition $\boldsymbol{c}_{ij} \in \operatorname{rng} \boldsymbol{\alpha}_{ij}^{123}$ translates to

  $$C_{ij} = \{(x_1, x_2, x_3) \in (\mathbb{Q}^+)^3 \mid (x_i, x_j) \neq (0,0) \text{ and } x_i^{-1} x_j \in U_{ij}\}$$

  for suitable U_{ij} .
- This way, the given conditions translate to the geometric conditions stated in "lattice Ceva".
- By the latter, we are lead to $\lambda, \mu \in \mathbb{Q}^{++}$ such that (up to identifications) $\boldsymbol{c}_{12} = \langle \lambda a \smallsetminus b \rangle$, $\boldsymbol{c}_{23} = \langle \mu b \smallsetminus c \rangle$, and $\boldsymbol{c}_{13} = \langle \lambda \mu a' \smallsetminus c \rangle$ (*recall that* $x \smallsetminus y = x - x \wedge y$).
- $\boldsymbol{c}_{13} \leq \boldsymbol{c}_{12} \vee \boldsymbol{c}_{23}$ yields $\langle \lambda \mu a' \smallsetminus c \rangle \leq \langle \lambda a \smallsetminus b \rangle \vee \langle \mu b \smallsetminus c \rangle$.

Idea of proof (cont'd)

- The condition $\boldsymbol{c}_{ij} \in \operatorname{rng} \boldsymbol{\alpha}_{ij}^{123}$ translates to

  $$C_{ij} = \{(x_1, x_2, x_3) \in (\mathbb{Q}^+)^3 \mid (x_i, x_j) \neq (0,0) \text{ and } x_i^{-1} x_j \in U_{ij}\}$$

  for suitable U_{ij} .
- This way, the given conditions translate to the geometric conditions stated in "lattice Ceva".
- By the latter, we are lead to $\lambda, \mu \in \mathbb{Q}^{++}$ such that (up to identifications) $\boldsymbol{c}_{12} = \langle \lambda a \smallsetminus b \rangle$, $\boldsymbol{c}_{23} = \langle \mu b \smallsetminus c \rangle$, and $\boldsymbol{c}_{13} = \langle \lambda \mu a' \smallsetminus c \rangle$ (*recall that* $x \smallsetminus y = x - x \wedge y$).
- $\boldsymbol{c}_{13} \leq \boldsymbol{c}_{12} \vee \boldsymbol{c}_{23}$ yields $\langle \lambda \mu a' \smallsetminus c \rangle \leq \langle \lambda a \smallsetminus b \rangle \vee \langle \mu b \smallsetminus c \rangle$.
- Letting $x \propto y$ hold if $(\exists k < \omega)(x \leq ky)$ and applying $f \colon A_{123} \to \mathbb{Q}$ sending (a, a', b, c) to $(1, 2, \lambda, \lambda\mu)$, we get $\lambda\mu = 2\lambda\mu \smallsetminus \lambda\mu \propto 0$, a contradiction.

Non-representability of the diagram $\mathrm{Id_c}\,\vec{A}$

Corollary

There are no $\mathfrak{P}[3]$-indexed commutative diagram $\vec{G}$, of (not necessarily Abelian) ℓ-groups, and no natural transformation $\eta\colon \mathrm{Cs_c}\,\vec{G} \twoheadrightarrow \mathrm{Id_c}\,\vec{A}$ with surjective arrows.

Non-representability of the diagram $\mathrm{Id_c}\,\vec{A}$

Corollary

There are no $\mathfrak{P}[3]$-indexed commutative diagram $\vec{G}$, of (not necessarily Abelian) ℓ-groups, and no natural transformation $\eta\colon \mathrm{Cs_c}\,\vec{G} \twoheadrightarrow \mathrm{Id_c}\,\vec{A}$ with surjective arrows.

Idea of proof.

For $i \in \{1,2,3\}$ pick $x_i \in G_i^+$ such that $\eta(\langle x_i \rangle_{G_i}) = \boldsymbol{a}_i$. Then the elements $\boldsymbol{c}_{ij} \overset{\text{def}}{=} \langle x_i \smallsetminus x_j \rangle_{G_{123}}$ satisfy (1)–(4) of previous lemma (e.g., $\boldsymbol{c}_{12} \wedge \boldsymbol{c}_{23} \leq \boldsymbol{c}_{13} \leq \boldsymbol{c}_{12} \vee \boldsymbol{c}_{23}$). $\square$

Non-representability of the diagram $\mathrm{Id_c}\,\vec{A}$

Corollary

There are no $\mathfrak{P}[3]$-indexed commutative diagram $\vec{G}$, of (not necessarily Abelian) ℓ-groups, and no natural transformation $\eta\colon \mathrm{Cs_c}\,\vec{G} \twoheadrightarrow \mathrm{Id_c}\,\vec{A}$ with surjective arrows.

Idea of proof.

For $i \in \{1,2,3\}$ pick $x_i \in G_i^+$ such that $\eta(\langle x_i \rangle_{G_i}) = \boldsymbol{a}_i$. Then the elements $\boldsymbol{c}_{ij} \overset{\text{def}}{=} \langle x_i \smallsetminus x_j \rangle_{G_{123}}$ satisfy (1)–(4) of previous lemma (e.g., $\boldsymbol{c}_{12} \wedge \boldsymbol{c}_{23} \leq \boldsymbol{c}_{13} \leq \boldsymbol{c}_{12} \vee \boldsymbol{c}_{23}$). $\square$

Corollary

There are no $\mathfrak{P}[3]$-indexed commutative diagram $\vec{G}$, of (not necessarily Abelian) ℓ-groups, such that $\mathrm{Cs_c}\,\vec{G} \cong \mathrm{Id_c}\,\vec{A}$ (resp., all G_p are representable and $\mathrm{Id_c}\,\vec{G} \cong \mathrm{Id_c}\,\vec{A}$).

The condensate construction (Gillibert and W 2011)

- For a poset P, a P-scaled Boolean algebra is a structure $\boldsymbol{B} = (B, (B^{(p)} \mid p \in P))$ where B is a Boolean algebra and each $B^{(p)}$ is an ideal of B, subjected to the conditions $1 \in \bigvee_p B^{(p)}$ and $B^{(p)} \cap B^{(q)} = \bigvee_{r \geq p,q} B^{(r)}$.

The condensate construction (Gillibert and W 2011)

- For a poset P, a P-scaled Boolean algebra is a structure $\boldsymbol{B} = (B, (B^{(p)} \mid p \in P))$ where B is a Boolean algebra and each $B^{(p)}$ is an ideal of B, subjected to the conditions $1 \in \bigvee_p B^{(p)}$ and $B^{(p)} \cap B^{(q)} = \bigvee_{r \geq p,q} B^{(r)}$.
- $\boldsymbol{B}$ is finitely presented iff B is finite and for every atom a of B, there exists a largest $p \in P$ such that $a \in B^{(p)}$; denote it by $|a|$.

The condensate construction (Gillibert and W 2011)

- For a poset P, a *P-scaled Boolean algebra* is a structure $\boldsymbol{B} = (B, (B^{(p)} \mid p \in P))$ where B is a Boolean algebra and each $B^{(p)}$ is an ideal of B, subjected to the conditions $1 \in \bigvee_p B^{(p)}$ and $B^{(p)} \cap B^{(q)} = \bigvee_{r \geq p,q} B^{(r)}$.
- $\boldsymbol{B}$ is finitely presented iff B is finite and for every atom a of B, there exists a largest $p \in P$ such that $a \in B^{(p)}$; denote it by $|a|$.
- If $\boldsymbol{B}$ is finitely presented, then, for a P-indexed diagram $\vec{S}$ (in any category $\mathcal{S}$ with finite products), set $\boldsymbol{B} \otimes \vec{S} \overset{\text{def}}{=} \prod_{a \in \operatorname{At} B} S_{|a|}$.

The condensate construction (Gillibert and W 2011)

- For a poset P, a P-scaled Boolean algebra is a structure $\boldsymbol{B} = (B, (B^{(p)} \mid p \in P))$ where B is a Boolean algebra and each $B^{(p)}$ is an ideal of B, subjected to the conditions $1 \in \bigvee_p B^{(p)}$ and $B^{(p)} \cap B^{(q)} = \bigvee_{r \geq p,q} B^{(r)}$.
- $\boldsymbol{B}$ is finitely presented iff B is finite and for every atom a of B, there exists a largest $p \in P$ such that $a \in B^{(p)}$; denote it by $|a|$.
- If $\boldsymbol{B}$ is finitely presented, then, for a P-indexed diagram $\vec{S}$ (in any category $\mathcal{S}$ with finite products), set $\boldsymbol{B} \otimes \vec{S} \stackrel{\mathrm{def}}{=} \prod_{a \in \mathrm{At}\, B} S_{|a|}$.
- Under suitable conditions on $\mathcal{S}$ and if $\vec{S}$ is commutative, this construction can be extended to arbitrary $\boldsymbol{B}$ by taking directed colimits. We say that $\boldsymbol{B} \otimes \vec{S}$ is a condensate of $\vec{S}$.

Towards the object counterexample

Main principle (Gillibert and W 2011)

If a commutative diagram $\vec{S}$ is a counterexample, at diagram level, to a representation problem (wrt. a given functor), then a suitable condensate $\mathbf{F}(X) \otimes \vec{S}$ is a counterexample to the same problem at object level.

Towards the object counterexample

Main principle (Gillibert and W 2011)

If a commutative diagram $\vec{S}$ is a counterexample, at diagram level, to a representation problem (wrt. a given functor), then a suitable condensate $\mathbf{F}(X) \otimes \vec{S}$ is a counterexample to the same problem at object level.

- For our current purposes, P is the cube $\mathfrak{P}[3]$ and $\mathbf{F}(X)$ is an explicitly constructed P-scaled Boolean algebra constructed from P.

Towards the object counterexample

Main principle (Gillibert and W 2011)

If a commutative diagram $\vec{S}$ is a counterexample, at diagram level, to a representation problem (wrt. a given functor), then a suitable condensate $\mathbf{F}(X) \otimes \vec{S}$ is a counterexample to the same problem at object level.

- For our current purposes, P is the cube $\mathfrak{P}[3]$ and $\mathbf{F}(X)$ is an explicitly constructed P-scaled Boolean algebra constructed from P.
- Due to the order-dimension of the cube being 3, the cardinality of $\mathbf{F}(X)$ needs to be pushed up to $\aleph_2$.

Towards the object counterexample

Main principle (Gillibert and W 2011)

If a commutative diagram $\vec{S}$ is a counterexample, at diagram level, to a representation problem (wrt. a given functor), then a suitable condensate $\mathbf{F}(X) \otimes \vec{S}$ is a counterexample to the same problem at object level.

- For our current purposes, P is the cube $\mathfrak{P}[3]$ and $\mathbf{F}(X)$ is an explicitly constructed P-scaled Boolean algebra constructed from P.
- Due to the order-dimension of the cube being 3, the cardinality of $\mathbf{F}(X)$ needs to be pushed up to $\aleph_2$.
- Here, there is no commutative diagram $\vec{G}$ of ℓ-groups such that $\mathrm{Cs_c}\,\vec{G} \cong \mathrm{Id_c}\,\vec{A}$.

The object counterexample

- It is $L \stackrel{\mathrm{def}}{=} \mathbf{F}(X) \otimes \mathrm{Id_c}\, \vec{A}$.

The object counterexample

- It is $L \stackrel{\mathrm{def}}{=} \mathbf{F}(X) \otimes \mathrm{Id_c}\,\vec{A}$.
- The cardinality of L is $\aleph_2$.

The object counterexample

- It is $L \overset{\mathrm{def}}{=} \mathbf{F}(X) \otimes \mathrm{Id_c}\, \vec{A}$.
- The cardinality of L is $\aleph_2$.
- By applying the "Armature Lemma" to the main negative property of $\vec{A}$, we obtain:

The object counterexample

- It is $L \overset{\mathrm{def}}{=} \mathbf{F}(X) \otimes \mathrm{Id_c}\, \vec{A}$.
- The cardinality of L is $\aleph_2$.
- By applying the "Armature Lemma" to the main negative property of $\vec{A}$, we obtain:

Theorem (W 2018)

The distributive lattice L it completely normal and has countably based differences. However, there are no ℓ-group G and no surjective homomorphism $\mathrm{Cs_c}\, G \twoheadrightarrow L$.

Cevian lattices

Definition

A binary operation $\smallsetminus$, on a distributive lattice D with zero, is Cevian if $x \leq y \vee (x \smallsetminus y)$, $(x \smallsetminus y) \wedge (y \smallsetminus x) = 0$, and $x \smallsetminus z \leq (x \smallsetminus y) \vee (y \smallsetminus z) \quad \forall x, y, z \in D$.

Cevian lattices

Definition

A binary operation $\smallsetminus$, on a distributive lattice D with zero, is Cevian if $x \leq y \vee (x \smallsetminus y)$, $(x \smallsetminus y) \wedge (y \smallsetminus x) = 0$, and $x \smallsetminus z \leq (x \smallsetminus y) \vee (y \smallsetminus z) \quad \forall x, y, z \in D$. We say that D is Cevian if it has a Cevian operation.

Cevian lattices

Definition

A binary operation $\smallsetminus$, on a distributive lattice D with zero, is Cevian if $x \leq y \vee (x \smallsetminus y)$, $(x \smallsetminus y) \wedge (y \smallsetminus x) = 0$, and $x \smallsetminus z \leq (x \smallsetminus y) \vee (y \smallsetminus z) \quad \forall x, y, z \in D$. We say that D is Cevian if it has a Cevian operation.

- Every Cevian operation satisfies the identity $(x \smallsetminus y) \wedge (y \smallsetminus z) \leq x \smallsetminus z$ (*Proof*: write $x \smallsetminus y \leq (x \smallsetminus z) \vee (z \smallsetminus y)$, then meet with $y \smallsetminus z$).

Cevian lattices

Definition

A binary operation $\smallsetminus$, on a distributive lattice D with zero, is Cevian if $x \leq y \vee (x \smallsetminus y)$, $(x \smallsetminus y) \wedge (y \smallsetminus x) = 0$, and $x \smallsetminus z \leq (x \smallsetminus y) \vee (y \smallsetminus z) \quad \forall x, y, z \in D$. We say that D is Cevian if it has a Cevian operation.

- Every Cevian operation satisfies the identity $(x \smallsetminus y) \wedge (y \smallsetminus z) \leq x \smallsetminus z$ (*Proof*: write $x \smallsetminus y \leq (x \smallsetminus z) \vee (z \smallsetminus y)$, then meet with $y \smallsetminus z$).
- The class of all Cevian lattices is closed under homomorphic images, products, ideals.

Cevian lattices

Definition

A binary operation $\smallsetminus$, on a distributive lattice D with zero, is Cevian if $x \leq y \vee (x \smallsetminus y)$, $(x \smallsetminus y) \wedge (y \smallsetminus x) = 0$, and $x \smallsetminus z \leq (x \smallsetminus y) \vee (y \smallsetminus z) \quad \forall x, y, z \in D$. We say that D is Cevian if it has a Cevian operation.

- Every Cevian operation satisfies the identity $(x \smallsetminus y) \wedge (y \smallsetminus z) \leq x \smallsetminus z$ (*Proof*: write $x \smallsetminus y \leq (x \smallsetminus z) \vee (z \smallsetminus y)$, then meet with $y \smallsetminus z$).
- The class of all Cevian lattices is closed under homomorphic images, products, ideals.
- For every (not necessarily Abelian) ℓ-group G, the lattice $\mathrm{Cs_c}\, G$ is Cevian (*Proof*: for each $\boldsymbol{x} \in \mathrm{Cs_c}\, G$ pick $\gamma(\boldsymbol{x})$ such that $\boldsymbol{x} = \langle \gamma(\boldsymbol{x}) \rangle$. Set $\boldsymbol{x} \smallsetminus \boldsymbol{y} \overset{\mathrm{def}}{=} \langle \gamma(\boldsymbol{x}) - \gamma(\boldsymbol{x}) \wedge \gamma(\boldsymbol{y}) \rangle$.)

Non-Cevian, glimpse at beyond

By inspecting the non-representability proof above, we get:

Non-Cevian, glimpse at beyond

By inspecting the non-representability proof above, we get:

Theorem (W 2018)

The lattice L is not Cevian.

Non-Cevian, glimpse at beyond

By inspecting the non-representability proof above, we get:

Theorem (W 2018)

The lattice L is not Cevian.

- Hence the implication Cevian $\Rightarrow$ completely normal cannot be reversed, even in the presence of countably based differences.

Non-Cevian, glimpse at beyond

By inspecting the non-representability proof above, we get:

Theorem (W 2018)

The lattice L is not Cevian.

- Hence the implication Cevian $\Rightarrow$ completely normal cannot be reversed, even in the presence of countably based differences.
- In particular, complete normality together with countably based differences are not sufficient to characterize the class of all $\mathrm{Id_c}\, G$ for Abelian ℓ-groups G.

Non-Cevian, glimpse at beyond

By inspecting the non-representability proof above, we get:

Theorem (W 2018)

The lattice L is not Cevian.

- Hence the implication Cevian $\Rightarrow$ completely normal cannot be reversed, even in the presence of countably based differences.
- In particular, complete normality together with countably based differences are not sufficient to characterize the class of all $\mathrm{Id}_c\, G$ for Abelian ℓ-groups G.
- How to extend this to arbitrary $\mathscr{L}_{\infty\lambda}$ sentences?

Non-Cevian, glimpse at beyond

By inspecting the non-representability proof above, we get:

Theorem (W 2018)

The lattice L is not Cevian.

- Hence the implication Cevian $\Rightarrow$ completely normal cannot be reversed, even in the presence of countably based differences.
- In particular, complete normality together with countably based differences are not sufficient to characterize the class of all $\mathrm{Id}_c\, G$ for Abelian ℓ-groups G.
- How to extend this to arbitrary $\mathscr{L}_{\infty\lambda}$ sentences?

The real trouble begins

Beyond

- The condensate construction $\boldsymbol{B} \otimes \vec{S}$ can be, under certain conditions, extended to the case where the diagram $\vec{S}$ is not commutative.

Beyond

- The condensate construction $\boldsymbol{B} \otimes \vec{S}$ can be, under certain conditions, extended to the case where the diagram $\vec{S}$ is not commutative.
- In that world, we define a similar object $\boldsymbol{B} \boxtimes \vec{S}$.

Beyond

- The condensate construction $\boldsymbol{B} \otimes \vec{S}$ can be, under certain conditions, extended to the case where the diagram $\vec{S}$ is not commutative.
- In that world, we define a similar object $\boldsymbol{B} \boxtimes \vec{S}$. This construction can no longer be extended by forming directed colimits (because $\vec{S}$ is not commutative).

Beyond

- The condensate construction $\boldsymbol{B} \otimes \vec{S}$ can be, under certain conditions, extended to the case where the diagram $\vec{S}$ is not commutative.
- In that world, we define a similar object $\boldsymbol{B} \boxtimes \vec{S}$. This construction can no longer be extended by forming directed colimits (because $\vec{S}$ is not commutative).
- However, in the ℓ-group context above ($\vec{S} := \vec{A}$), $\mathrm{Id_c}\, \vec{A}^I$ is a commutative diagram $\forall I$.

Beyond

- The condensate construction $\boldsymbol{B} \otimes \vec{S}$ can be, under certain conditions, extended to the case where the diagram $\vec{S}$ is not commutative.
- In that world, we define a similar object $\boldsymbol{B} \boxtimes \vec{S}$. This construction can no longer be extended by forming directed colimits (because $\vec{S}$ is not commutative).
- However, in the ℓ-group context above ($\vec{S} := \vec{A}$), $\mathrm{Id_c}\,\vec{A}^I$ is a commutative diagram $\forall I$.
- It follows that for every infinite regular cardinal λ, we can extend the construction $\boldsymbol{B} \mapsto \mathrm{Id_c}(\boldsymbol{B} \boxtimes \vec{A})$ to arbitrary $\boldsymbol{B}$, now by forming λ-directed colimits.

Beyond

- The condensate construction $\boldsymbol{B} \otimes \vec{S}$ can be, under certain conditions, extended to the case where the diagram $\vec{S}$ is not commutative.
- In that world, we define a similar object $\boldsymbol{B} \boxtimes \vec{S}$. This construction can no longer be extended by forming directed colimits (because $\vec{S}$ is not commutative).
- However, in the ℓ-group context above ($\vec{S} := \vec{A}$), $\mathrm{Id_c}\, \vec{A}^I$ is a commutative diagram $\forall I$.
- It follows that for every infinite regular cardinal λ, we can extend the construction $\boldsymbol{B} \mapsto \mathrm{Id_c}(\boldsymbol{B} \boxtimes \vec{A})$ to arbitrary $\boldsymbol{B}$, now by forming λ-directed colimits.
- A fairly large amount of machinery needs to be set up.

After the dust settles

- We introduce a purely categorical property of a morphism, called anti-elementarity.

After the dust settles

- We introduce a purely categorical property of a morphism, called anti-elementarity.
- If a category $\mathcal{C}$ of models is anti-elementary, then for every infinite cardinal λ there are models A, B such that A is an $\mathscr{L}_{\infty\lambda}$-elementary submodel of B, $A \in \mathcal{C}$, and $B \notin \mathcal{C}$.

After the dust settles

- We introduce a purely categorical property of a morphism, called anti-elementarity.
- If a category $\mathcal{C}$ of models is anti-elementary, then for every infinite cardinal λ there are models A, B such that A is an $\mathscr{L}_{\infty\lambda}$-elementary submodel of B, $A \in \mathcal{C}$, and $B \notin \mathcal{C}$.
- In particular, an anti-elementary category cannot be the class of all models of any class of $\mathscr{L}_{\infty\lambda}$ sentences.

After the dust settles

- We introduce a purely categorical property of a morphism, called anti-elementarity.
- If a category $\mathcal{C}$ of models is anti-elementary, then for every infinite cardinal λ there are models A, B such that A is an $\mathscr{L}_{\infty\lambda}$-elementary submodel of B, $A \in \mathcal{C}$, and $B \notin \mathcal{C}$.
- In particular, an anti-elementary category cannot be the class of all models of any class of $\mathscr{L}_{\infty\lambda}$ sentences.

Theorem (W 2018)

Let $\mathcal{G}$ be a class of (not necessarily commutative) ℓ-groups.

1. If $\mathcal{G}$ contains all Archimedean ℓ-groups, then $\mathrm{Cs_c}\,\mathcal{G}$ is anti-elementary.
2. If $\mathcal{G}$ is a nontrivial quasivariety of ℓ-groups, then $\mathrm{Id_c}\,\mathcal{G}$ is anti-elementary.

Anti-elementarity unleashed

- In the above (case where not all members of $\mathcal{G}$ are representable), the proof of the $\mathrm{Id_c}$ part involves further non-commutative diagrams of ℓ-groups (*arising from earlier research on CLP*).

Anti-elementarity unleashed

- In the above (case where not all members of $\mathcal{G}$ are representable), the proof of the $\mathrm{Id_c}$ part involves further non-commutative diagrams of ℓ-groups (*arising from earlier research on CLP*).
- The techniques above can be used to establish anti-elementarity of various other classes:
  1. Semilattices of finitely generated two-sided ideals in (von Neumann regular) rings.
  2. Semilattices of finitely generated submodules of modules.
  3. Nonstable K_0-theory of von Neumann regular rings (resp., C*-algebras of real rank zero).
  4. Coordinatizable lattices with a large 4-frame but without unit (*requires large cardinals so far*).

Thanks for your attention!