Spectral spaces of countable Abelian ℓ-groups

Friedrich Wehrung

LMNO, CNRS UMR 6139 (Caen)
E-mail: friedrich.wehrung01@unicaen.fr
URL: http://www.math.unicaen.fr/~wehrung

May 2017

The ℓ-spectrum of an Abelian ℓ-group

- An ℓ-subgroup I, in an Abelian ℓ-group G, is an ℓ-ideal if it is order-convex.

The ℓ-spectrum of an Abelian ℓ-group

- An ℓ-subgroup I, in an Abelian ℓ-group G, is an ℓ-ideal if it is order-convex.
- An ℓ-ideal I is prime if $I \neq G$ and $x \wedge y \in I$ implies that either $x \in I$ or $y \in I$ $(\forall x, y \in G)$.

The ℓ-spectrum of an Abelian ℓ-group

- An ℓ-subgroup I, in an Abelian ℓ-group G, is an ℓ-ideal if it is order-convex.
- An ℓ-ideal I is prime if $I \neq G$ and $x \wedge y \in I$ implies that either $x \in I$ or $y \in I$ $(\forall x, y \in G)$.
- We endow the set $\mathrm{Spec}_\ell\, G$, of all prime ℓ-ideals of G, with the topology whose closed sets are exactly the $V_G(X) \underset{\text{def}}{=} \{P \in \mathrm{Spec}_\ell\, G \mid X \subseteq P\}$, for $X \subseteq G$.

The ℓ-spectrum of an Abelian ℓ-group

- An ℓ-subgroup I, in an Abelian ℓ-group G, is an ℓ-ideal if it is order-convex.
- An ℓ-ideal I is prime if $I \neq G$ and $x \wedge y \in I$ implies that either $x \in I$ or $y \in I$ $(\forall x, y \in G)$.
- We endow the set $\mathrm{Spec}_\ell\, G$, of all prime ℓ-ideals of G, with the topology whose closed sets are exactly the $V_G(X) \underset{\mathrm{def}}{=} \{P \in \mathrm{Spec}_\ell\, G \mid X \subseteq P\}$, for $X \subseteq G$.
- The topological space $\mathrm{Spec}_\ell\, G$ is called the ℓ-spectrum of G.

The ℓ-spectrum of an Abelian ℓ-group

- An ℓ-subgroup I, in an Abelian ℓ-group G, is an ℓ-ideal if it is order-convex.
- An ℓ-ideal I is prime if $I \neq G$ and $x \wedge y \in I$ implies that either $x \in I$ or $y \in I$ $(\forall x, y \in G)$.
- We endow the set $\mathrm{Spec}_\ell\, G$, of all prime ℓ-ideals of G, with the topology whose closed sets are exactly the $V_G(X) \underset{\text{def}}{=} \{P \in \mathrm{Spec}_\ell\, G \mid X \subseteq P\}$, for $X \subseteq G$.
- The topological space $\mathrm{Spec}_\ell\, G$ is called the ℓ-spectrum of G.

Problem ('90s)

Characterize the topological spaces of the form $\mathrm{Spec}_\ell\, G$, for Abelian ℓ-groups G.

The ℓ-spectrum of an Abelian ℓ-group

- An ℓ-subgroup I, in an Abelian ℓ-group G, is an ℓ-ideal if it is order-convex.
- An ℓ-ideal I is prime if $I \neq G$ and $x \wedge y \in I$ implies that either $x \in I$ or $y \in I$ $(\forall x, y \in G)$.
- We endow the set $\mathrm{Spec}_\ell\, G$, of all prime ℓ-ideals of G, with the topology whose closed sets are exactly the $V_G(X) \underset{\mathrm{def}}{=} \{P \in \mathrm{Spec}_\ell\, G \mid X \subseteq P\}$, for $X \subseteq G$.
- The topological space $\mathrm{Spec}_\ell\, G$ is called the ℓ-spectrum of G.

Problem ('90s)

Characterize the topological spaces of the form $\mathrm{Spec}_\ell\, G$, for Abelian ℓ-groups G.

Equivalent formulation: describe the spectra of MV-algebras.

Spectrum of a distributive lattice with zero

- An ideal, in a distributive lattice D with zero, is a nonempty lower subset closed under $(x, y) \mapsto x \vee y$.

Spectrum of a distributive lattice with zero

- An ideal, in a distributive lattice D with zero, is a nonempty lower subset closed under $(x, y) \mapsto x \vee y$.
- An ideal I is prime if $I \neq D$ and $x \wedge y \in I$ implies that either $x \in I$ or $y \in I$ $(\forall x, y \in D)$.

Spectrum of a distributive lattice with zero

- An ideal, in a distributive lattice D with zero, is a nonempty lower subset closed under $(x, y) \mapsto x \vee y$.
- An ideal I is prime if $I \neq D$ and $x \wedge y \in I$ implies that either $x \in I$ or $y \in I$ $(\forall x, y \in D)$.
- We endow the set $\mathrm{Spec}\, D$, of all prime ideals of D, with the topology whose closed sets are exactly the $V_D(X) \underset{\mathrm{def}}{=} \{P \in \mathrm{Spec}\, D \mid X \subseteq P\}$, for $X \subseteq D$.

Spectrum of a distributive lattice with zero

- An ideal, in a distributive lattice D with zero, is a nonempty lower subset closed under $(x, y) \mapsto x \vee y$.
- An ideal I is prime if $I \neq D$ and $x \wedge y \in I$ implies that either $x \in I$ or $y \in I$ $(\forall x, y \in D)$.
- We endow the set $\mathrm{Spec}\, D$, of all prime ideals of D, with the topology whose closed sets are exactly the $V_D(X) \underset{\mathrm{def}}{=} \{P \in \mathrm{Spec}\, D \mid X \subseteq P\}$, for $X \subseteq D$.
- The topological space $\mathrm{Spec}\, D$ is called the spectrum of D.

Spectrum of a distributive lattice with zero

- An ideal, in a distributive lattice D with zero, is a nonempty lower subset closed under $(x, y) \mapsto x \vee y$.
- An ideal I is prime if $I \neq D$ and $x \wedge y \in I$ implies that either $x \in I$ or $y \in I$ $(\forall x, y \in D)$.
- We endow the set $\mathrm{Spec}\, D$, of all prime ideals of D, with the topology whose closed sets are exactly the $V_D(X) \underset{\mathrm{def}}{=} \{P \in \mathrm{Spec}\, D \mid X \subseteq P\}$, for $X \subseteq D$.
- The topological space $\mathrm{Spec}\, D$ is called the spectrum of D.

Stone duality ('30s)

The assignments $D \mapsto \mathrm{Spec}\, D$, and $X \mapsto \mathcal{K}(X) \underset{\mathrm{def}}{=}$ lattice of all compact open subsets of X, define a duality between distributive lattices with zero and sober spaces in which the compact open subsets form a basis of the topology, closed under $(X, Y) \mapsto X \cap Y$ (generalized spectral spaces).

The lattice $\mathrm{Id_c}\, G$

- Every finitely generated ℓ-ideal, in an Abelian ℓ-group G, is generated by a single element of G^+ (for $\langle a_1, \dots, a_n \rangle = \langle |a_1| \vee \cdots \vee |a_n| \rangle\ \forall a_1, \dots, a_n \in G$).

The lattice $\mathrm{Id_c}\, G$

- Every finitely generated ℓ-ideal, in an Abelian ℓ-group G, is generated by a single element of G^+ (for $\langle a_1, \dots, a_n \rangle = \langle |a_1| \vee \cdots \vee |a_n| \rangle \ \forall a_1, \dots, a_n \in G$).
- $\langle a \rangle \vee \langle b \rangle = \langle a \vee b \rangle = \langle a + b \rangle$ and $\langle a \rangle \cap \langle b \rangle = \langle a \wedge b \rangle$, for all $a, b \in G^+$.

The lattice $\operatorname{Id_c} G$

- Every finitely generated ℓ-ideal, in an Abelian ℓ-group G, is generated by a single element of G^+ (for $\langle a_1, \dots, a_n \rangle = \langle |a_1| \vee \cdots \vee |a_n| \rangle \ \forall a_1, \dots, a_n \in G$).
- $\langle a \rangle \vee \langle b \rangle = \langle a \vee b \rangle = \langle a + b \rangle$ and $\langle a \rangle \cap \langle b \rangle = \langle a \wedge b \rangle$, for all $a, b \in G^+$.
- Hence, $\operatorname{Id_c} G \underset{\text{def}}{=} \{\langle a \rangle \mid a \in G^+\}$ is a distributive lattice with zero. Call such lattices ℓ-representable.

The lattice $\operatorname{Id}_c G$

- Every finitely generated ℓ-ideal, in an Abelian ℓ-group G, is generated by a single element of G^+ (for $\langle a_1, \dots, a_n \rangle = \langle |a_1| \vee \cdots \vee |a_n| \rangle\ \forall a_1, \dots, a_n \in G$).
- $\langle a \rangle \vee \langle b \rangle = \langle a \vee b \rangle = \langle a + b \rangle$ and $\langle a \rangle \cap \langle b \rangle = \langle a \wedge b \rangle$, for all $a, b \in G^+$.
- Hence, $\operatorname{Id}_c G \underset{\mathrm{def}}{=} \{\langle a \rangle \mid a \in G^+\}$ is a distributive lattice with zero. Call such lattices ℓ-representable.
- For every ℓ-ideal I of the ℓ-group G, $\varphi(I) \underset{\mathrm{def}}{=} \{\langle x \rangle \mid x \in I\}$ is an ideal of the lattice $\operatorname{Id}_c G$.

The lattice $\mathrm{Id_c}\, G$

- Every finitely generated ℓ-ideal, in an Abelian ℓ-group G, is generated by a single element of G^+ (for $\langle a_1, \dots, a_n \rangle = \langle |a_1| \vee \cdots \vee |a_n| \rangle\ \forall a_1, \dots, a_n \in G$).
- $\langle a \rangle \vee \langle b \rangle = \langle a \vee b \rangle = \langle a + b \rangle$ and $\langle a \rangle \cap \langle b \rangle = \langle a \wedge b \rangle$, for all $a, b \in G^+$.
- Hence, $\mathrm{Id_c}\, G \underset{\mathrm{def}}{=} \{\langle a \rangle \mid a \in G^+\}$ is a distributive lattice with zero. Call such lattices ℓ-representable.
- For every ℓ-ideal I of the ℓ-group G, $\varphi(I) \underset{\mathrm{def}}{=} \{\langle x \rangle \mid x \in I\}$ is an ideal of the lattice $\mathrm{Id_c}\, G$.
- For every ideal $\boldsymbol{I}$ of the lattice $\mathrm{Id_c}\, G$, $\psi(\boldsymbol{I}) \underset{\mathrm{def}}{=} \{x \in G \mid \langle x \rangle \in \boldsymbol{I}\}$ is an ℓ-ideal of the ℓ-group G.

The lattice $\mathrm{Id_c}\, G$

- Every finitely generated ℓ-ideal, in an Abelian ℓ-group G, is generated by a single element of G^+ (for $\langle a_1, \dots, a_n \rangle = \langle |a_1| \vee \cdots \vee |a_n| \rangle\ \forall a_1, \dots, a_n \in G$).
- $\langle a \rangle \vee \langle b \rangle = \langle a \vee b \rangle = \langle a + b \rangle$ and $\langle a \rangle \cap \langle b \rangle = \langle a \wedge b \rangle$, for all $a, b \in G^+$.
- Hence, $\mathrm{Id_c}\, G \underset{\mathrm{def}}{=} \{\langle a \rangle \mid a \in G^+\}$ is a distributive lattice with zero. Call such lattices ℓ-representable.
- For every ℓ-ideal I of the ℓ-group G, $\varphi(I) \underset{\mathrm{def}}{=} \{\langle x \rangle \mid x \in I\}$ is an ideal of the lattice $\mathrm{Id_c}\, G$.
- For every ideal $\boldsymbol{I}$ of the lattice $\mathrm{Id_c}\, G$, $\psi(\boldsymbol{I}) \underset{\mathrm{def}}{=} \{x \in G \mid \langle x \rangle \in \boldsymbol{I}\}$ is an ℓ-ideal of the ℓ-group G.
- φ and ψ are mutually inverse, and they both preserve primeness.

The lattice $\mathrm{Id_c}\, G$

- Every finitely generated ℓ-ideal, in an Abelian ℓ-group G, is generated by a single element of G^+ (for $\langle a_1, \dots, a_n \rangle = \langle |a_1| \vee \cdots \vee |a_n| \rangle\ \forall a_1, \dots, a_n \in G$).
- $\langle a \rangle \vee \langle b \rangle = \langle a \vee b \rangle = \langle a + b \rangle$ and $\langle a \rangle \cap \langle b \rangle = \langle a \wedge b \rangle$, for all $a, b \in G^+$.
- Hence, $\mathrm{Id_c}\, G \underset{\mathrm{def}}{=} \{\langle a \rangle \mid a \in G^+\}$ is a distributive lattice with zero. Call such lattices ℓ-representable.
- For every ℓ-ideal I of the ℓ-group G, $\varphi(I) \underset{\mathrm{def}}{=} \{\langle x \rangle \mid x \in I\}$ is an ideal of the lattice $\mathrm{Id_c}\, G$.
- For every ideal $\boldsymbol{I}$ of the lattice $\mathrm{Id_c}\, G$, $\psi(\boldsymbol{I}) \underset{\mathrm{def}}{=} \{x \in G \mid \langle x \rangle \in \boldsymbol{I}\}$ is an ℓ-ideal of the ℓ-group G.
- φ and ψ are mutually inverse, and they both preserve primeness.
- Hence, $\mathrm{Spec}_\ell\, G \cong \mathrm{Spec}\, \mathrm{Id_c}\, G$, so it is also a generalized spectral space.

The lattice $\mathrm{Id_c}\, G$

- Every finitely generated ℓ-ideal, in an Abelian ℓ-group G, is generated by a single element of G^+ (for $\langle a_1, \dots, a_n \rangle = \langle |a_1| \vee \cdots \vee |a_n| \rangle\ \forall a_1, \dots, a_n \in G$).
- $\langle a \rangle \vee \langle b \rangle = \langle a \vee b \rangle = \langle a + b \rangle$ and $\langle a \rangle \cap \langle b \rangle = \langle a \wedge b \rangle$, for all $a, b \in G^+$.
- Hence, $\mathrm{Id_c}\, G \underset{\text{def}}{=} \{\langle a \rangle \mid a \in G^+\}$ is a distributive lattice with zero. Call such lattices ℓ-representable.
- For every ℓ-ideal I of the ℓ-group G, $\varphi(I) \underset{\text{def}}{=} \{\langle x \rangle \mid x \in I\}$ is an ideal of the lattice $\mathrm{Id_c}\, G$.
- For every ideal $\boldsymbol{I}$ of the lattice $\mathrm{Id_c}\, G$, $\psi(\boldsymbol{I}) \underset{\text{def}}{=} \{x \in G \mid \langle x \rangle \in \boldsymbol{I}\}$ is an ℓ-ideal of the ℓ-group G.
- φ and ψ are mutually inverse, and they both preserve primeness.
- Hence, $\mathrm{Spec}_\ell\, G \cong \mathrm{Spec}\,\mathrm{Id_c}\, G$, so it is also a generalized spectral space.
- Hence, $\mathrm{Spec}_\ell\, G$ and $\mathrm{Id_c}\, G$ determine each other (*via* Stone duality).

The lattice $\mathrm{Id_c}\, G$

- Every finitely generated ℓ-ideal, in an Abelian ℓ-group G, is generated by a single element of G^+ (for $\langle a_1, \dots, a_n \rangle = \langle |a_1| \vee \cdots \vee |a_n| \rangle\ \forall a_1, \dots, a_n \in G$).
- $\langle a \rangle \vee \langle b \rangle = \langle a \vee b \rangle = \langle a + b \rangle$ and $\langle a \rangle \cap \langle b \rangle = \langle a \wedge b \rangle$, for all $a, b \in G^+$.
- Hence, $\mathrm{Id_c}\, G \underset{\mathrm{def}}{=} \{\langle a \rangle \mid a \in G^+\}$ is a distributive lattice with zero. Call such lattices ℓ-representable.
- For every ℓ-ideal I of the ℓ-group G, $\varphi(I) \underset{\mathrm{def}}{=} \{\langle x \rangle \mid x \in I\}$ is an ideal of the lattice $\mathrm{Id_c}\, G$.
- For every ideal $\boldsymbol{I}$ of the lattice $\mathrm{Id_c}\, G$, $\psi(\boldsymbol{I}) \underset{\mathrm{def}}{=} \{x \in G \mid \langle x \rangle \in \boldsymbol{I}\}$ is an ℓ-ideal of the ℓ-group G.
- φ and ψ are mutually inverse, and they both preserve primeness.
- Hence, $\mathrm{Spec}_\ell\, G \cong \mathrm{Spec}\,\mathrm{Id_c}\, G$, so it is also a generalized spectral space.
- Hence, $\mathrm{Spec}_\ell\, G$ and $\mathrm{Id_c}\, G$ determine each other (*via* Stone duality).
- Recasts the above problem as: Describe ℓ-representable lattices.

Complete normality

- Specialization order on a T_0 space: $x \leqslant y$ if $y \in \mathrm{cl}\,\{x\}$.

Complete normality

- Specialization order on a T_0 space: $x \leqslant y$ if $y \in \mathrm{cl}\,\{x\}$.
- A generalized spectral space X is completely normal if its specialization order is a root system, that is, $\forall x, y, z \in X$, if $\{x, y\} \subseteq \mathrm{cl}\,\{z\}$, then $x \in \mathrm{cl}\,\{y\}$ or $y \in \mathrm{cl}\,\{x\}$. This holds if (not iff) every subspace of X is normal in the usual sense.

Complete normality

- Specialization order on a T_0 space: $x \leqslant y$ if $y \in \mathrm{cl}\,\{x\}$.
- A generalized spectral space X is completely normal if its specialization order is a root system, that is, $\forall x, y, z \in X$, if $\{x, y\} \subseteq \mathrm{cl}\,\{z\}$, then $x \in \mathrm{cl}\,\{y\}$ or $y \in \mathrm{cl}\,\{x\}$. This holds if (not iff) every subspace of X is normal in the usual sense.
- A distributive lattice D with zero is completely normal if $\forall a, b \in D$, $\exists x, y \in D$ such that $a \leq b \vee x$, $b \leq a \vee y$, and $x \wedge y = 0$.

Complete normality

- Specialization order on a T_0 space: $x \leqslant y$ if $y \in \mathrm{cl}\,\{x\}$.
- A generalized spectral space X is completely normal if its specialization order is a root system, that is, $\forall x, y, z \in X$, if $\{x, y\} \subseteq \mathrm{cl}\,\{z\}$, then $x \in \mathrm{cl}\,\{y\}$ or $y \in \mathrm{cl}\,\{x\}$. This holds if (not iff) every subspace of X is normal in the usual sense.
- A distributive lattice D with zero is completely normal if $\forall a, b \in D$, $\exists x, y \in D$ such that $a \leq b \vee x$, $b \leq a \vee y$, and $x \wedge y = 0$ (we say that (x, y) is a splitting of (a, b)).

Complete normality

- Specialization order on a T_0 space: $x \leqslant y$ if $y \in \mathrm{cl}\,\{x\}$.
- A generalized spectral space X is completely normal if its specialization order is a root system, that is, $\forall x, y, z \in X$, if $\{x, y\} \subseteq \mathrm{cl}\,\{z\}$, then $x \in \mathrm{cl}\,\{y\}$ or $y \in \mathrm{cl}\,\{x\}$. This holds if (not iff) every subspace of X is normal in the usual sense.
- A distributive lattice D with zero is completely normal if $\forall a, b \in D$, $\exists x, y \in D$ such that $a \leq b \vee x$, $b \leq a \vee y$, and $x \wedge y = 0$ (we say that (x, y) is a splitting of (a, b)).

Theorem (Monteiro 1956)

A generalized spectral space X is completely normal iff the distributive lattice $\mathcal{K}(X)$ is completely normal.

Complete normality of $\mathrm{Id_c}\, G$

Proposition (folklore)

For every Abelian ℓ-group G, $\mathrm{Id_c}\, G$ is a completely normal distributive lattice (equivalently, $\mathrm{Spec}_\ell\, G$ is a completely normal generalized spectral space).

Complete normality of $\operatorname{Id_c} G$

Proposition (folklore)

For every Abelian ℓ-group G, $\operatorname{Id_c} G$ is a completely normal distributive lattice (equivalently, $\operatorname{Spec}_\ell G$ is a completely normal generalized spectral space).

Proof.

Let $\boldsymbol{a}, \boldsymbol{b} \in \operatorname{Id_c} G$. There are $a, b \in G^+$ such that $\boldsymbol{a} = \langle a \rangle$ and $\boldsymbol{b} = \langle b \rangle$. Set $\boldsymbol{x} \underset{\text{def}}{=} \langle a - a \wedge b \rangle$ and $\boldsymbol{y} \underset{\text{def}}{=} \langle b - a \wedge b \rangle$. Then $(\boldsymbol{x}, \boldsymbol{y})$ is a splitting of $(\boldsymbol{a}, \boldsymbol{b})$. □

Countably based differences

Definition

A distributive lattice D has countably based differences if $\forall a, b \in D$, the set $a \ominus b \underset{\text{def}}{=} \{x \in D \mid a \leq x \vee b\}$ has a countable coinitial subset.

Countably based differences

Definition

A distributive lattice D has countably based differences if $\forall a, b \in D$, the set $a \ominus b \underset{\text{def}}{=} \{x \in D \mid a \leq x \vee b\}$ has a countable coinitial subset.

(i.e., $\{c_n \mid n < \omega\} \subseteq a \ominus b$ such that $\forall x \in a \ominus b\ \exists n < \omega\ c_n \leq x$)

Countably based differences

Definition

A distributive lattice D has countably based differences if $\forall a, b \in D$, the set $a \ominus b \underset{\text{def}}{=} \{x \in D \mid a \leq x \vee b\}$ has a countable coinitial subset.

(i.e., $\{c_n \mid n < \omega\} \subseteq a \ominus b$ such that $\forall x \in a \ominus b\ \exists n < \omega\ c_n \leq x$)

Proposition (Cignoli, Gluschankof, and Lucas 1999)

Let G be an Abelian ℓ-group. Then $\mathrm{Id_c}\, G$ has countably based differences.

Countably based differences

Definition

A distributive lattice D has countably based differences if $\forall a, b \in D$, the set $a \ominus b \underset{\text{def}}{=} \{x \in D \mid a \leq x \vee b\}$ has a countable coinitial subset.

(i.e., $\{c_n \mid n < \omega\} \subseteq a \ominus b$ such that $\forall x \in a \ominus b\ \exists n < \omega\ c_n \leq x$)

Proposition (Cignoli, Gluschankof, and Lucas 1999)

Let G be an Abelian ℓ-group. Then $\mathrm{Id_c}\, G$ has countably based differences.

Proof.

If $\boldsymbol{a} = \langle a \rangle$ and $\boldsymbol{b} = \langle b \rangle$ (where $a, b \in G^+$), set $\boldsymbol{c}_n \underset{\text{def}}{=} \langle a - a \wedge nb \rangle$.

Then $\{\boldsymbol{c}_n \mid n < \omega\}$ is coinitial in $\boldsymbol{a} \ominus \boldsymbol{b}$. □

Non-ℓ-representable lattices

Theorem (Delzell and Madden, 1994)

There exists a non-ℓ-representable bounded distributive lattice of cardinality $\aleph_1$.

Non-ℓ-representable lattices

Theorem (Delzell and Madden, 1994)

There exists a non-ℓ-representable bounded distributive lattice of cardinality $\aleph_1$.

- Delzell and Madden also have a much more complicated example of a completely normal spectral space which is not the real spectrum of any commutative, unital ring.

Non-ℓ-representable lattices

Theorem (Delzell and Madden, 1994)

There exists a non-ℓ-representable bounded distributive lattice of cardinality $\aleph_1$.

- Delzell and Madden also have a much more complicated example of a completely normal spectral space which is not the real spectrum of any commutative, unital ring.
- This example is not second countable either.

No $\mathscr{L}_{\infty,\omega}$ characterization of ℓ-representability

- Set $\boldsymbol{B}_I \underset{\mathrm{def}}{=} \{X \subseteq I \mid X \text{ or } I \setminus X \text{ is finite}\}$ and

$$\boldsymbol{D}_I \underset{\mathrm{def}}{=} \{(X,k) \in \boldsymbol{B}_I \times \{0,1,2\} \mid (k=0 \Rightarrow X \text{ finite}) \text{ and } (k \neq 0 \Rightarrow I \setminus X \text{ finite})\}\,,$$

for any set I.

No $\mathscr{L}_{\infty,\omega}$ characterization of ℓ-representability

- Set $\boldsymbol{B}_I \underset{\mathrm{def}}{=} \{X \subseteq I \mid X \text{ or } I \setminus X \text{ is finite}\}$ and

$$\boldsymbol{D}_I \underset{\mathrm{def}}{=} \{(X,k) \in \boldsymbol{B}_I \times \{0,1,2\} \mid (k=0 \Rightarrow X \text{ finite}) \text{ and } (k \neq 0 \Rightarrow I \setminus X \text{ finite})\}\,,$$

  for any set I.
- $\boldsymbol{D}_\omega \hookrightarrow \boldsymbol{D}_{\omega_1}$, *via*

$$(X,k) \mapsto \begin{cases} (X,k)\,, & \text{if } k=0\,, \\ (X \cup (\omega_1 \setminus \omega), k)\,, & \text{if } k \neq 0\,. \end{cases}$$

No $\mathscr{L}_{\infty,\omega}$ characterization of ℓ-representability

- Set $\boldsymbol{B}_I \underset{\mathrm{def}}{=} \{X \subseteq I \mid X \text{ or } I \setminus X \text{ is finite}\}$ and

$$\boldsymbol{D}_I \underset{\mathrm{def}}{=} \{(X,k) \in \boldsymbol{B}_I \times \{0,1,2\} \mid (k=0 \Rightarrow X \text{ finite}) \text{ and } (k \neq 0 \Rightarrow I \setminus X \text{ finite})\},$$

  for any set I.

- $\boldsymbol{D}_\omega \hookrightarrow \boldsymbol{D}_{\omega_1}$, *via*

$$(X,k) \mapsto \begin{cases} (X,k)\,, & \text{if } k=0\,, \\ (X \cup (\omega_1 \setminus \omega), k)\,, & \text{if } k \neq 0\,. \end{cases}$$

Proposition (W 2017)

$\boldsymbol{D}_\omega$ is an $\mathscr{L}_{\infty,\omega}$-elementary sublattice of $\boldsymbol{D}_{\omega_1}$ (use back-and-forth), with $\boldsymbol{D}_\omega$ countable (and ℓ-representable) and $\boldsymbol{D}_{\omega_1}$ non-ℓ-representable (no countably based differences).

No $\mathscr{L}_{\infty,\omega}$ characterization of ℓ-representability

- Set $\boldsymbol{B}_I \underset{\text{def}}{=} \{X \subseteq I \mid X \text{ or } I \setminus X \text{ is finite}\}$ and

$$\boldsymbol{D}_I \underset{\text{def}}{=} \{(X,k) \in \boldsymbol{B}_I \times \{0,1,2\} \mid (k = 0 \Rightarrow X \text{ finite}) \text{ and } (k \neq 0 \Rightarrow I \setminus X \text{ finite})\}\,,$$

  for any set I.
- $\boldsymbol{D}_\omega \hookrightarrow \boldsymbol{D}_{\omega_1}$, *via*

$$(X,k) \mapsto \begin{cases} (X,k)\,, & \text{if } k = 0\,,\\ (X \cup (\omega_1 \setminus \omega), k)\,, & \text{if } k \neq 0\,. \end{cases}$$

Proposition (W 2017)

$\boldsymbol{D}_\omega$ is an $\mathscr{L}_{\infty,\omega}$-elementary sublattice of $\boldsymbol{D}_{\omega_1}$ (use back-and-forth), with $\boldsymbol{D}_\omega$ countable (and ℓ-representable) and $\boldsymbol{D}_{\omega_1}$ non-ℓ-representable (no countably based differences). Consequently, ℓ-representability is not $\mathscr{L}_{\infty,\omega}$-definable.

Generalized dual Heyting algebras

Definition

A distributive lattice D with zero is a generalized dual Heyting algebra if $\forall a, b \in D$, $\exists$ smallest $x \in D$ such that $a \leq b \vee x$;

Generalized dual Heyting algebras

Definition

A distributive lattice D with zero is a generalized dual Heyting algebra if $\forall a, b \in D$, $\exists$ smallest $x \in D$ such that $a \leq b \vee x$; then denoted by $a \smallsetminus_D b$ and called the pseudo-difference of a and b.

Generalized dual Heyting algebras

Definition

A distributive lattice D with zero is a generalized dual Heyting algebra if $\forall a, b \in D$, $\exists$ smallest $x \in D$ such that $a \leq b \vee x$; then denoted by $a \smallsetminus_D b$ and called the pseudo-difference of a and b.

Theorem (Cignoli, Gluschankof, and Lucas 1999)

Every dual generalized Heyting algebra is ℓ-representable.

Generalized dual Heyting algebras

Definition

A distributive lattice D with zero is a generalized dual Heyting algebra if $\forall a, b \in D$, $\exists$ smallest $x \in D$ such that $a \leq b \vee x$; then denoted by $a \smallsetminus_D b$ and called the pseudo-difference of a and b.

Theorem (Cignoli, Gluschankof, and Lucas 1999)

Every dual generalized Heyting algebra is ℓ-representable.

The proof extends (non-trivially) the finite case.

Generalized dual Heyting algebras

Definition

A distributive lattice D with zero is a generalized dual Heyting algebra if $\forall a, b \in D$, $\exists$ smallest $x \in D$ such that $a \leq b \vee x$; then denoted by $a \smallsetminus_D b$ and called the pseudo-difference of a and b.

Theorem (Cignoli, Gluschankof, and Lucas 1999)

Every dual generalized Heyting algebra is ℓ-representable.

The proof extends (non-trivially) the finite case. In that case, D is the lattice of all lower subsets of a finite root system P.

Generalized dual Heyting algebras

Definition

A distributive lattice D with zero is a generalized dual Heyting algebra if $\forall a, b \in D$, $\exists$ smallest $x \in D$ such that $a \leq b \vee x$; then denoted by $a \smallsetminus_D b$ and called the pseudo-difference of a and b.

Theorem (Cignoli, Gluschankof, and Lucas 1999)

Every dual generalized Heyting algebra is ℓ-representable.

The proof extends (non-trivially) the finite case. In that case, D is the lattice of all lower subsets of a finite root system P. So $D \cong \mathrm{Id_c}\, \mathbb{Q}\langle P \rangle$, where $\mathbb{Q}\langle P \rangle$ is the lexicographical power (Hahn power) of $\mathbb{Q}$ by P.

Closed lattice homomorphisms

Definition

For distributive lattices D and E with zero, a 0-lattice homomorphism $f\colon D \to E$ is closed if for all $a, b \in D$ and all $c \in E$, $f(a) \leq f(b) \vee c \Rightarrow \exists x \in D$, $a \leq b \vee x$ and $f(x) \leq c$.

Closed lattice homomorphisms

Definition

For distributive lattices D and E with zero, a 0-lattice homomorphism $f\colon D \to E$ is closed if for all $a, b \in D$ and all $c \in E$, $f(a) \leq f(b) \vee c \Rightarrow \exists x \in D,\ a \leq b \vee x$ and $f(x) \leq c$.

Equivalently, the dual map $\mathrm{Spec}\, f\colon\ \mathrm{Spec}\, E \to \mathrm{Spec}\, D$ sends closed subsets to closed subsets (resp., sends upper subsets to upper subsets).

Closed lattice homomorphisms

Definition

For distributive lattices D and E with zero, a 0-lattice homomorphism $f\colon D \to E$ is closed if for all $a, b \in D$ and all $c \in E$, $f(a) \leq f(b) \vee c \Rightarrow \exists x \in D$, $a \leq b \vee x$ and $f(x) \leq c$.

Equivalently, the dual map $\mathrm{Spec}\, f\colon\ \mathrm{Spec}\, E \to \mathrm{Spec}\, D$ sends closed subsets to closed subsets (resp., sends upper subsets to upper subsets).

Proposition

Let $f\colon G \to H$ be a ℓ-homomorphism between Abelian ℓ-groups. Then $\mathrm{Id_c}\, f\colon\ \mathrm{Id_c}\, G \to \mathrm{Id_c}\, H$ is a closed lattice homomorphism.

Closed lattice homomorphisms

Definition

For distributive lattices D and E with zero, a 0-lattice homomorphism $f\colon D \to E$ is closed if for all $a, b \in D$ and all $c \in E$, $f(a) \leq f(b) \vee c \Rightarrow \exists x \in D,\ a \leq b \vee x$ and $f(x) \leq c$.

Equivalently, the dual map $\mathrm{Spec}\, f\colon\ \mathrm{Spec}\, E \to \mathrm{Spec}\, D$ sends closed subsets to closed subsets (resp., sends upper subsets to upper subsets).

Proposition

Let $f\colon G \to H$ be a ℓ-homomorphism between Abelian ℓ-groups. Then $\mathrm{Id_c}\, f\colon\ \mathrm{Id_c}\, G \to \mathrm{Id_c}\, H$ is a closed lattice homomorphism.

Proposition

Let G be an Abelian ℓ-group, let $\boldsymbol{D}$ be a distributive lattice with zero. Then every surjective closed lattice homomorphism $\boldsymbol{f}\colon\ \mathrm{Id_c}\, G \twoheadrightarrow \boldsymbol{D}$ induces an isomorphism $\mathrm{Id_c}\,(G/I) \to \boldsymbol{D}$, for the ℓ-ideal $I \underset{\text{def}}{=} \{x \in G \mid \boldsymbol{f}(\langle x \rangle) = 0\}$.

A positive result

The aim of what follows is to sketch a proof of the following result:

A positive result

The aim of what follows is to sketch a proof of the following result:

Theorem (W 2017)

Every countable, completely normal distributive lattice with zero is ℓ-representable.

A positive result

The aim of what follows is to sketch a proof of the following result:

Theorem (W 2017)

Every countable, completely normal distributive lattice with zero is ℓ-representable.

Equivalently (using Stone duality and Monteiro's result),

A positive result

The aim of what follows is to sketch a proof of the following result:

Theorem (W 2017)

Every countable, completely normal distributive lattice with zero is ℓ-representable.

Equivalently (using Stone duality and Monteiro's result),

Every second countable, completely normal generalized spectral space is the ℓ-spectrum of some Abelian ℓ-group

A positive result

The aim of what follows is to sketch a proof of the following result:

Theorem (W 2017)

Every countable, completely normal distributive lattice with zero is ℓ-representable.

Equivalently (using Stone duality and Monteiro's result),

Every second countable, completely normal generalized spectral space is the ℓ-spectrum of some Abelian ℓ-group

Strategy: starting with a countable, completely normal distributive lattice D with zero, we construct an ascending tower of lattice homomorphisms $f_n \colon E_n \to D$, where $\bigcup_{n<\omega} E_n = \mathrm{Id_c}\, \mathrm{F}_\ell(\omega)$, with suitably chosen finite E_n and failures of closedness / surjectivity / being defined everywhere corrected at each stage.

A positive result

The aim of what follows is to sketch a proof of the following result:

Theorem (W 2017)

Every countable, completely normal distributive lattice with zero is ℓ-representable.

Equivalently (using Stone duality and Monteiro's result),

Every second countable, completely normal generalized spectral space is the ℓ-spectrum of some Abelian ℓ-group

Strategy: starting with a countable, completely normal distributive lattice D with zero, we construct an ascending tower of lattice homomorphisms $f_n \colon E_n \to D$, where $\bigcup_{n<\omega} E_n = \mathrm{Id_c}\, \mathrm{F}_\ell(\omega)$, with suitably chosen finite E_n and failures of closedness / surjectivity / being defined everywhere corrected at each stage.
A 2004 example by Di Nola and Grigolia shows that the E_n cannot always be taken completely normal.

Defining $\mathrm{Op}(\mathcal{H})$

Definition

Let $\mathcal{H}$ be a set of closed hyperplanes in a topological vector space $\mathbb{E}$ over $\mathbb{R}$.

Defining $\mathrm{Op}(\mathcal{H})$

Definition

Let $\mathcal{H}$ be a set of closed hyperplanes in a topological vector space $\mathbb{E}$ over $\mathbb{R}$. We set

$$\mathrm{Bool}(\mathcal{H}) \underset{\text{def}}{=} \text{Boolean subalgebra of the powerset of } \mathbb{E}$$

$$\text{generated by all } H^+ \text{ and } H^-\,, \text{ where } H \in \mathcal{H}\,;$$

$$\mathrm{Op}(\mathcal{H}) \underset{\text{def}}{=} \{\text{open members of } \mathrm{Bool}(\mathcal{H})\}\,.$$

(The E_n will have the form $\mathrm{Op}^-(\mathcal{H}) \underset{\text{def}}{=} \mathrm{Op}(\mathcal{H}) \setminus \{\mathbb{E}\}$.)

Defining $\mathrm{Op}(\mathcal{H})$

Definition

Let $\mathcal{H}$ be a set of closed hyperplanes in a topological vector space $\mathbb{E}$ over $\mathbb{R}$. We set

$$\mathrm{Bool}(\mathcal{H}) \underset{\mathrm{def}}{=} \text{Boolean subalgebra of the powerset of } \mathbb{E}$$

$$\text{generated by all } H^+ \text{ and } H^-, \text{ where } H \in \mathcal{H};$$

$$\mathrm{Op}(\mathcal{H}) \underset{\mathrm{def}}{=} \{\text{open members of } \mathrm{Bool}(\mathcal{H})\}.$$

(The E_n will have the form $\mathrm{Op}^-(\mathcal{H}) \underset{\mathrm{def}}{=} \mathrm{Op}(\mathcal{H}) \setminus \{\mathbb{E}\}$.)

Lemma

For every $X \in \mathrm{Bool}(\mathcal{H})$, $\mathrm{int}(X)$ belongs to $\mathrm{Op}(\mathcal{H})$, and it is a finite union of sets of the form $\bigcap_{i=1}^{n} H_i^{\pm}$, where all $H_i \in \mathcal{H}$ (basic open sets).

Defining $\mathrm{Op}(\mathcal{H})$

Definition

Let $\mathcal{H}$ be a set of closed hyperplanes in a topological vector space $\mathbb{E}$ over $\mathbb{R}$. We set

$$\mathrm{Bool}(\mathcal{H}) \underset{\mathrm{def}}{=} \text{Boolean subalgebra of the powerset of } \mathbb{E}$$
$$\text{generated by all } H^+ \text{ and } H^-, \text{ where } H \in \mathcal{H}\,;$$
$$\mathrm{Op}(\mathcal{H}) \underset{\mathrm{def}}{=} \{\text{open members of } \mathrm{Bool}(\mathcal{H})\}\,.$$

(The E_n will have the form $\mathrm{Op}^-(\mathcal{H}) \underset{\mathrm{def}}{=} \mathrm{Op}(\mathcal{H}) \setminus \{\mathbb{E}\}$.)

Lemma

For every $X \in \mathrm{Bool}(\mathcal{H})$, $\mathrm{int}(X)$ belongs to $\mathrm{Op}(\mathcal{H})$, and it is a finite union of sets of the form $\bigcap_{i=1}^{n} H_i^{\pm}$, where all $H_i \in \mathcal{H}$ (basic open sets). Moreover, $\mathrm{Op}(\mathcal{H})$ is a Heyting subalgebra of the algebra of all open subsets of $\mathbb{E}$.

The join-irreducibles of Op($\mathcal{H}$)

Let $\mathcal{H}$ be a nonempty finite set of closed hyperplanes in a topological vector space $\mathbb{E}$.

The join-irreducibles of $\mathrm{Op}(\mathcal{H})$

Let $\mathcal{H}$ be a nonempty finite set of closed hyperplanes in a topological vector space $\mathbb{E}$.

Notation

For $U \in \mathrm{Op}(\mathcal{H})$, we set

$$\mathcal{H}_U \underset{\mathrm{def}}{=} \{H \in \mathcal{H} \mid H \cap U \neq \varnothing\}\ ,$$

$$\nabla_{\mathcal{H}} U = \nabla U \underset{\mathrm{def}}{=} \text{intersection of all members of } \mathcal{H}_U\ .$$

The join-irreducibles of $\mathrm{Op}(\mathcal{H})$

Let $\mathcal{H}$ be a nonempty finite set of closed hyperplanes in a topological vector space $\mathbb{E}$.

Notation

For $U \in \mathrm{Op}(\mathcal{H})$, we set

$$\mathcal{H}_U \underset{\mathrm{def}}{=} \{H \in \mathcal{H} \mid H \cap U \neq \varnothing\}\,,$$

$$\nabla_{\mathcal{H}} U = \nabla U \underset{\mathrm{def}}{=} \text{intersection of all members of } \mathcal{H}_U\,.$$

Thus, ∇U is a closed subspace of $\mathbb{E}$, with finite codimension.

Characterizing the join-irreducibles

By the above, every join-irreducible member of $\mathrm{Op}(\mathcal{H})$ is convex.

Characterizing the join-irreducibles

By the above, every join-irreducible member of $\mathrm{Op}(\mathcal{H})$ is convex.

Lemma

A convex member P of $\mathrm{Op}(\mathcal{H})$ is join-irreducible iff $P \cap \nabla P \neq \varnothing$, in which case $P_* = P \setminus \nabla P$ and $P^\dagger = \complement\big(\mathrm{cl}(P) \cap \nabla P\big) = \complement\, \mathrm{cl}(P \cap \nabla P)$ (the largest $X \in \mathrm{Op}(\mathcal{H})$ such that $P \not\subseteq X$).

Characterizing the join-irreducibles

By the above, every join-irreducible member of $\mathrm{Op}(\mathcal{H})$ is convex.

Lemma

A convex member P of $\mathrm{Op}(\mathcal{H})$ is join-irreducible iff $P \cap \nabla P \neq \varnothing$, in which case $P_* = P \setminus \nabla P$ and $P^\dagger = \complement(\mathrm{cl}(P) \cap \nabla P) = \complement\,\mathrm{cl}(P \cap \nabla P)$ (the largest $X \in \mathrm{Op}(\mathcal{H})$ such that $P \not\subseteq X$).

- Recall that in any finite distributive lattice D, $p \mapsto p^\dagger$ is an order-isomorphism between $\mathrm{Ji}\, D \underset{\text{def}}{=} \{\text{join-irreducibles of } D\}$ and $\mathrm{Mi}\, D \underset{\text{def}}{=} \{\text{meet-irreducibles of } D\}$ (with induced $\leq$ from D).

Characterizing the join-irreducibles

By the above, every join-irreducible member of $\mathrm{Op}(\mathcal{H})$ is convex.

Lemma

A convex member P of $\mathrm{Op}(\mathcal{H})$ is join-irreducible iff $P \cap \nabla P \neq \varnothing$, in which case $P_* = P \setminus \nabla P$ and $P^\dagger = \complement\big(\mathrm{cl}(P) \cap \nabla P\big) = \complement\,\mathrm{cl}(P \cap \nabla P)$ (the largest $X \in \mathrm{Op}(\mathcal{H})$ such that $P \nsubseteq X$).

- Recall that in any finite distributive lattice D, $p \mapsto p^\dagger$ is an order-isomorphism between $\mathrm{Ji}\, D \underset{\mathrm{def}}{=} \{\text{join-irreducibles of } D\}$ and $\mathrm{Mi}\, D \underset{\mathrm{def}}{=} \{\text{meet-irreducibles of } D\}$ (with induced $\leq$ from D).
- Important observation about $\mathrm{Op}(\mathcal{H})$: $P \setminus P_* = P \cap \nabla P$ is convex $\forall P \in \mathrm{Ji}\,\mathrm{Op}(\mathcal{H})$.

Characterizing the join-irreducibles

By the above, every join-irreducible member of $\mathrm{Op}(\mathcal{H})$ is convex.

Lemma

A convex member P of $\mathrm{Op}(\mathcal{H})$ is join-irreducible iff $P \cap \nabla P \neq \varnothing$, in which case $P_* = P \setminus \nabla P$ and $P^\dagger = \complement(\mathrm{cl}(P) \cap \nabla P) = \complement\,\mathrm{cl}(P \cap \nabla P)$ (the largest $X \in \mathrm{Op}(\mathcal{H})$ such that $P \not\subseteq X$).

- Recall that in any finite distributive lattice D, $p \mapsto p^\dagger$ is an order-isomorphism between $\mathrm{Ji}\, D \underset{\mathrm{def}}{=} \{\text{join-irreducibles of } D\}$ and $\mathrm{Mi}\, D \underset{\mathrm{def}}{=} \{\text{meet-irreducibles of } D\}$ (with induced $\leq$ from D).
- Important observation about $\mathrm{Op}(\mathcal{H})$: $P \setminus P_* = P \cap \nabla P$ is convex $\forall P \in \mathrm{Ji}\,\mathrm{Op}(\mathcal{H})$.

Corollary

Let P and Q be join-irreducibles in $\mathrm{Op}(\mathcal{H})$. Then $P \subsetneq Q$ implies $\nabla Q \subsetneq \nabla P$.

Consonance

Definition

Let D be a distributive lattice with zero. Elements $a, b \in D$ are consonant, in notation $a \sim b$, if $\exists x, y \in D$ such that $a \leq b \vee x$, $b \leq a \vee y$, and $x \wedge y = 0$ (again: we say that (x, y) is a splitting of (a, b)).

Consonance

Definition

Let D be a distributive lattice with zero. Elements $a, b \in D$ are consonant, in notation $a \sim b$, if $\exists x, y \in D$ such that $a \leq b \vee x$, $b \leq a \vee y$, and $x \wedge y = 0$ (again: we say that (x, y) is a splitting of (a, b)).

In particular, D is completely normal iff any two elements of D are consonant (i.e., D is a consonant subset of itself).

Consonance

Definition

Let D be a distributive lattice with zero. Elements $a, b \in D$ are consonant, in notation $a \sim b$, if $\exists x, y \in D$ such that $a \leq b \vee x$, $b \leq a \vee y$, and $x \wedge y = 0$ (again: we say that (x, y) is a splitting of (a, b)).

In particular, D is completely normal iff any two elements of D are consonant (i.e., D is a consonant subset of itself).

Lemma

1. $a \leq b \Rightarrow a \sim b$;
2. $a \sim b \Rightarrow b \sim a$;
3. $(a \sim c \text{ and } b \sim c) \Rightarrow (a \vee b \sim c \text{ and } a \wedge b \sim c)$.

Difference operations

Definition

Let L be a lattice and let S be a lattice with zero. A map $L \times L \to S$, $(x, y) \mapsto x \smallsetminus y$ is a difference operation if

Difference operations

Definition

Let L be a lattice and let S be a lattice with zero. A map $L \times L \to S$, $(x, y) \mapsto x \smallsetminus y$ is a difference operation if

1. $x \smallsetminus x = 0$, $\forall x \in L$;
2. $x \smallsetminus z = (x \smallsetminus y) \vee (y \smallsetminus z)$, whenever $x \geq y \geq z$ in L;
3. $x \smallsetminus y = (x \vee y) \smallsetminus y = x \smallsetminus (x \wedge y)$, $\forall x, y \in L$.

Difference operations

Definition

Let L be a lattice and let S be a lattice with zero. A map $L \times L \to S$, $(x, y) \mapsto x \smallsetminus y$ is a difference operation if

1. $x \smallsetminus x = 0$, $\forall x \in L$;
2. $x \smallsetminus z = (x \smallsetminus y) \vee (y \smallsetminus z)$, whenever $x \geq y \geq z$ in L;
3. $x \smallsetminus y = (x \vee y) \smallsetminus y = x \smallsetminus (x \wedge y)$, $\forall x, y \in L$.

It is a normal difference operation if $(x \smallsetminus y) \wedge (y \smallsetminus x) = 0$ $\forall x, y \in L$.

Difference operations

Definition

Let L be a lattice and let S be a lattice with zero. A map $L \times L \to S$, $(x, y) \mapsto x \smallsetminus y$ is a difference operation if

1. $x \smallsetminus x = 0$, $\forall x \in L$;
2. $x \smallsetminus z = (x \smallsetminus y) \vee (y \smallsetminus z)$, whenever $x \geq y \geq z$ in L;
3. $x \smallsetminus y = (x \vee y) \smallsetminus y = x \smallsetminus (x \wedge y)$, $\forall x, y \in L$.

It is a normal difference operation if $(x \smallsetminus y) \wedge (y \smallsetminus x) = 0$ $\forall x, y \in L$.

Lemma (*Triangle Inequality*)

$x \smallsetminus z \leq (x \smallsetminus y) \vee (y \smallsetminus z)$, $\forall x, y, z \in L$.

Difference operations

Definition

Let L be a lattice and let S be a lattice with zero. A map $L \times L \to S$, $(x, y) \mapsto x \smallsetminus y$ is a difference operation if

1. $x \smallsetminus x = 0$, $\forall x \in L$;
2. $x \smallsetminus z = (x \smallsetminus y) \vee (y \smallsetminus z)$, whenever $x \geq y \geq z$ in L;
3. $x \smallsetminus y = (x \vee y) \smallsetminus y = x \smallsetminus (x \wedge y)$, $\forall x, y \in L$.

It is a normal difference operation if $(x \smallsetminus y) \wedge (y \smallsetminus x) = 0$ $\forall x, y \in L$.

Lemma (*Triangle Inequality*)

$x \smallsetminus z \leq (x \smallsetminus y) \vee (y \smallsetminus z)$, $\forall x, y, z \in L$.

Lemma

Let L be finite. Then $a \smallsetminus b = \bigvee (p \smallsetminus p_* \mid p \in \mathrm{Ji}\, L\,,\ p \leq a\,,\ p \nleq b)$, $\forall a, b \in L$.

Pseudo-differences again

Lemma

Let D be a finite distributive lattice. Then the pseudo-difference, $(x, y) \mapsto x \smallsetminus_D y \underset{\text{def}}{=}$ least $z \in D$ such that $x \leq y \vee z$, is a D-valued difference operation on D, normal on every consonant sublattice of D.

Pseudo-differences again

Lemma

Let D be a finite distributive lattice. Then the pseudo-difference, $(x, y) \mapsto x \smallsetminus_D y \underset{\text{def}}{=}$ least $z \in D$ such that $x \leq y \vee z$, is a D-valued difference operation on D, normal on every consonant sublattice of D.

The two following lemmas are crucial to further computations.

Pseudo-differences again

Lemma

Let D be a finite distributive lattice. Then the pseudo-difference, $(x, y) \mapsto x \smallsetminus_D y \underset{\text{def}}{=}$ least $z \in D$ such that $x \leq y \vee z$, is a D-valued difference operation on D, normal on every consonant sublattice of D.

The two following lemmas are crucial to further computations.

Lemma

Let D be a finite distributive lattice and let $a_1, a_2, b \in D$. Then

Pseudo-differences again

Lemma

Let D be a finite distributive lattice. Then the pseudo-difference, $(x, y) \mapsto x \smallsetminus_D y \underset{\text{def}}{=}$ least $z \in D$ such that $x \leq y \vee z$, is a D-valued difference operation on D, normal on every consonant sublattice of D.

The two following lemmas are crucial to further computations.

Lemma

Let D be a finite distributive lattice and let $a_1, a_2, b \in D$. Then

1. $(a_1 \vee a_2) \smallsetminus_D b = (a_1 \smallsetminus_D b) \vee (a_2 \smallsetminus_D b)$;
2. if $a_1 \sim a_2$, then $(a_1 \wedge a_2) \smallsetminus_D b = (a_1 \smallsetminus_D b) \wedge (a_2 \smallsetminus_D b)$;
3. the dual statements $(\leq \leftrightharpoons \geq)$ hold.

Pseudo-differences again

Lemma

Let D be a finite distributive lattice. Then the pseudo-difference, $(x, y) \mapsto x \smallsetminus_D y \underset{\text{def}}{=} \text{least } z \in D$ such that $x \leq y \vee z$, is a D-valued difference operation on D, normal on every consonant sublattice of D.

The two following lemmas are crucial to further computations.

Lemma

Let D be a finite distributive lattice and let $a_1, a_2, b \in D$. Then

1. $(a_1 \vee a_2) \smallsetminus_D b = (a_1 \smallsetminus_D b) \vee (a_2 \smallsetminus_D b)$;
2. if $a_1 \sim a_2$, then $(a_1 \wedge a_2) \smallsetminus_D b = (a_1 \smallsetminus_D b) \wedge (a_2 \smallsetminus_D b)$;
3. the dual statements $(\leq \leftrightharpoons \geq)$ hold.

Lemma

If $a_1 \sim a_2$ and $a_1 \wedge a_2 \leq b_1 \wedge b_2$, then $(a_1 \smallsetminus_D b_1) \wedge (a_2 \smallsetminus_D b_2) = 0$.

The Extension Lemma

Problem: we are given finite distributive lattices E and L, a $0,1$-sublattice D of E, and a 0-lattice homomorphism $f\colon D \to L$.

The Extension Lemma

Problem: we are given finite distributive lattices E and L, a $0,1$-sublattice D of E, and a 0-lattice homomorphism $f: D \to L$. Find a sufficient condition for f to have an extension to a lattice homomorphism $g: E \to L$.

The Extension Lemma

Problem: we are given finite distributive lattices E and L, a $0,1$-sublattice D of E, and a 0-lattice homomorphism $f\colon D \to L$. Find a sufficient condition for f to have an extension to a lattice homomorphism $g\colon E \to L$.

Extension Lemma for lattices

Suppose that there are $a, b \in E$ such that the following statements hold:

The Extension Lemma

Problem: we are given finite distributive lattices E and L, a $0,1$-sublattice D of E, and a 0-lattice homomorphism $f\colon D \to L$. Find a sufficient condition for f to have an extension to a lattice homomorphism $g\colon E \to L$.

Extension Lemma for lattices

Suppose that there are $a, b \in E$ such that the following statements hold:

1. (The range of) f is consonant in L;
2. $E = D[a, b]$;
3. D is a Heyting subalgebra of E;
4. $a \wedge b = 0$;
5. $\forall p \in \mathrm{Ji}\, D,\ p \leq p_* \vee a \vee b \Rightarrow (p \leq p_* \vee a \text{ or } p \leq p_* \vee b)$;
6. $\forall p, q \in \mathrm{Ji}\, D,\ (p \leq p_* \vee a \text{ and } q \leq q_* \vee b) \Rightarrow (p \text{ and } q \text{ are incomparable})$.

The Extension Lemma

Problem: we are given finite distributive lattices E and L, a $0,1$-sublattice D of E, and a 0-lattice homomorphism $f\colon D \to L$. Find a sufficient condition for f to have an extension to a lattice homomorphism $g\colon E \to L$.

Extension Lemma for lattices

Suppose that there are $a, b \in E$ such that the following statements hold:

1. (The range of) f is consonant in L;
2. $E = D[a, b]$;
3. D is a Heyting subalgebra of E;
4. $a \wedge b = 0$;
5. $\forall p \in \mathrm{Ji}\, D,\ p \leq p_* \vee a \vee b \Rightarrow (p \leq p_* \vee a \text{ or } p \leq p_* \vee b)$;
6. $\forall p, q \in \mathrm{Ji}\, D,\ (p \leq p_* \vee a \text{ and } q \leq q_* \vee b) \Rightarrow (p \text{ and } q \text{ are incomparable})$.

Then such an extension g exists, with $g(a) = f_*(a)$ and $g(b) = f_*(b)$, where $f_*(t) = \bigvee (f(p) \smallsetminus_L f(p_*) \mid p \in \mathrm{Ji}\, D\,,\ p \leq p_* \vee t)$, $\forall t \in E$.

The Extension Lemma for $\mathrm{Op}(\mathcal{H})$

Extension Lemma for $\mathrm{Op}(\mathcal{H})$

Let $\mathcal{H}$ be a finite set of closed hyperplanes in a topological vector space $\mathbb{E}$, let H be a closed hyperplane of $\mathbb{E}$, and let L be a finite distributive lattice.

The Extension Lemma for $\mathrm{Op}(\mathcal{H})$

Extension Lemma for $\mathrm{Op}(\mathcal{H})$

Let $\mathcal{H}$ be a finite set of closed hyperplanes in a topological vector space $\mathbb{E}$, let H be a closed hyperplane of $\mathbb{E}$, and let L be a finite distributive lattice. Then every consonant 0-lattice homomorphism $f\colon \mathrm{Op}(\mathcal{H}) \to L$ can be extended to a unique lattice homomorphism $g\colon \mathrm{Op}(\mathcal{H} \cup \{H\}) \to L$ such that $g(H^{\pm}) = f_*(H^{\pm})$, where

The Extension Lemma for $\mathrm{Op}(\mathcal{H})$

Extension Lemma for $\mathrm{Op}(\mathcal{H})$

Let $\mathcal{H}$ be a finite set of closed hyperplanes in a topological vector space $\mathbb{E}$, let H be a closed hyperplane of $\mathbb{E}$, and let L be a finite distributive lattice. Then every consonant 0-lattice homomorphism $f\colon \mathrm{Op}(\mathcal{H}) \to L$ can be extended to a unique lattice homomorphism $g\colon \mathrm{Op}(\mathcal{H} \cup \{H\}) \to L$ such that $g(H^{\pm}) = f_*(H^{\pm})$, where

$$f_*(U) \underset{\mathrm{def}}{=} \bigvee \left(f(P) \smallsetminus_L f(P_*) \mid P \in \mathrm{Ji}\, D\,,\ P \cap \nabla P \subseteq U \right)\,,\ \forall U \in \mathrm{Op}(\mathcal{H})\,.$$

The Extension Lemma for $\mathrm{Op}(\mathcal{H})$

Extension Lemma for $\mathrm{Op}(\mathcal{H})$

Let $\mathcal{H}$ be a finite set of closed hyperplanes in a topological vector space $\mathbb{E}$, let H be a closed hyperplane of $\mathbb{E}$, and let L be a finite distributive lattice. Then every consonant 0-lattice homomorphism $f\colon \mathrm{Op}(\mathcal{H}) \to L$ can be extended to a unique lattice homomorphism $g\colon \mathrm{Op}(\mathcal{H} \cup \{H\}) \to L$ such that $g(H^{\pm}) = f_*(H^{\pm})$, where

$$f_*(U) \underset{\mathrm{def}}{=} \bigvee \left(f(P) \smallsetminus_L f(P_*) \mid P \in \mathrm{Ji}\, D\,,\ P \cap \nabla P \subseteq U\right)\,,\ \forall U \in \mathrm{Op}(\mathcal{H})\,.$$

Outline of proof. Verify one by one the conditions of the Extension Lemma for lattices, with $D := \mathrm{Op}(\mathcal{H})$, $E := \mathrm{Op}(\mathcal{H} \cup \{H\})$, $a := H^+$, and $b := H^-$.

The Extension Lemma for $\mathrm{Op}(\mathcal{H})$

Extension Lemma for $\mathrm{Op}(\mathcal{H})$

Let $\mathcal{H}$ be a finite set of closed hyperplanes in a topological vector space $\mathbb{E}$, let H be a closed hyperplane of $\mathbb{E}$, and let L be a finite distributive lattice. Then every consonant 0-lattice homomorphism $f\colon \mathrm{Op}(\mathcal{H}) \to L$ can be extended to a unique lattice homomorphism $g\colon \mathrm{Op}(\mathcal{H} \cup \{H\}) \to L$ such that $g(H^{\pm}) = f_*(H^{\pm})$, where

$$f_*(U) \underset{\text{def}}{=} \bigvee \left(f(P) \smallsetminus_L f(P_*) \mid P \in \mathrm{Ji}\, D\,,\ P \cap \nabla P \subseteq U\right)\,,\ \forall U \in \mathrm{Op}(\mathcal{H})\,.$$

Outline of proof. Verify one by one the conditions of the Extension Lemma for lattices, with $D := \mathrm{Op}(\mathcal{H})$, $E := \mathrm{Op}(\mathcal{H} \cup \{H\})$, $a := H^+$, and $b := H^-$.

- Every basic open set in $\mathrm{Op}(\mathcal{H} \cup \{H\})$ has the form U or $U \cap H^{\pm}$, where U is basic open in $\mathrm{Op}(\mathcal{H})$; whence $E = D[a, b]$.

Extension Lemma for Op($\mathcal{H}$) (cont'd)

- Both $D = \mathrm{Op}(\mathcal{H})$ and $E = \mathrm{Op}(\mathcal{H} \cup \{H\})$ are Heyting subalgebras of the lattice of all open subsets of $\mathbb{E}$; whence D is a Heyting subalgebra of E.

Extension Lemma for $\mathrm{Op}(\mathcal{H})$ (cont'd)

- Both $D = \mathrm{Op}(\mathcal{H})$ and $E = \mathrm{Op}(\mathcal{H} \cup \{H\})$ are Heyting subalgebras of the lattice of all open subsets of $\mathbb{E}$; whence D is a Heyting subalgebra of E.
- Condition (4) now. Let $P \subseteq P_* \cup H^+ \cup H^-$, that is, $P \cap \nabla P \subseteq H^+ \cup H^-$.

Extension Lemma for $\mathrm{Op}(\mathcal{H})$ (cont'd)

- Both $D = \mathrm{Op}(\mathcal{H})$ and $E = \mathrm{Op}(\mathcal{H} \cup \{H\})$ are Heyting subalgebras of the lattice of all open subsets of $\mathbb{E}$; whence D is a Heyting subalgebra of E.
- Condition (4) now. Let $P \subseteq P_* \cup H^+ \cup H^-$, that is, $P \cap \nabla P \subseteq H^+ \cup H^-$.
- Since $P \cap \nabla P$ is convex, either $P \cap \nabla P \subseteq H^+$ or $P \cap \nabla P \subseteq H^-$, that is, either $P \subseteq P_* \cup H^+$ or $P \subseteq P_* \cup H^-$.

Extension Lemma for $\mathrm{Op}(\mathcal{H})$ (cont'd)

- Both $D = \mathrm{Op}(\mathcal{H})$ and $E = \mathrm{Op}(\mathcal{H} \cup \{H\})$ are Heyting subalgebras of the lattice of all open subsets of $\mathbb{E}$; whence D is a Heyting subalgebra of E.
- Condition (4) now. Let $P \subseteq P_* \cup H^+ \cup H^-$, that is, $P \cap \nabla P \subseteq H^+ \cup H^-$.
- Since $P \cap \nabla P$ is convex, either $P \cap \nabla P \subseteq H^+$ or $P \cap \nabla P \subseteq H^-$, that is, either $P \subseteq P_* \cup H^+$ or $P \subseteq P_* \cup H^-$.
- Condition (5) now. Let $P \cap \nabla P \subseteq H^+$ and $Q \cap \nabla Q \subseteq H^-$. Suppose, by way of contradiction, that $P \subseteq Q$.

Extension Lemma for $\mathrm{Op}(\mathcal{H})$ (cont'd)

- Both $D = \mathrm{Op}(\mathcal{H})$ and $E = \mathrm{Op}(\mathcal{H} \cup \{H\})$ are Heyting subalgebras of the lattice of all open subsets of $\mathbb{E}$; whence D is a Heyting subalgebra of E.
- Condition (4) now. Let $P \subseteq P_* \cup H^+ \cup H^-$, that is, $P \cap \nabla P \subseteq H^+ \cup H^-$.
- Since $P \cap \nabla P$ is convex, either $P \cap \nabla P \subseteq H^+$ or $P \cap \nabla P \subseteq H^-$, that is, either $P \subseteq P_* \cup H^+$ or $P \subseteq P_* \cup H^-$.
- Condition (5) now. Let $P \cap \nabla P \subseteq H^+$ and $Q \cap \nabla Q \subseteq H^-$. Suppose, by way of contradiction, that $P \subseteq Q$.
- Then $P^\dagger \subseteq Q^\dagger$, so $\mathrm{cl}(Q \cap \nabla Q) \subseteq \mathrm{cl}(P \cap \nabla P) \subseteq \overline{H}^+$.

Extension Lemma for $\operatorname{Op}(\mathcal{H})$ (cont'd)

- Both $D = \operatorname{Op}(\mathcal{H})$ and $E = \operatorname{Op}(\mathcal{H} \cup \{H\})$ are Heyting subalgebras of the lattice of all open subsets of $\mathbb{E}$; whence D is a Heyting subalgebra of E.
- Condition (4) now. Let $P \subseteq P_* \cup H^+ \cup H^-$, that is, $P \cap \nabla P \subseteq H^+ \cup H^-$.
- Since $P \cap \nabla P$ is convex, either $P \cap \nabla P \subseteq H^+$ or $P \cap \nabla P \subseteq H^-$, that is, either $P \subseteq P_* \cup H^+$ or $P \subseteq P_* \cup H^-$.
- Condition (5) now. Let $P \cap \nabla P \subseteq H^+$ and $Q \cap \nabla Q \subseteq H^-$. Suppose, by way of contradiction, that $P \subseteq Q$.
- Then $P^\dagger \subseteq Q^\dagger$, so $\operatorname{cl}(Q \cap \nabla Q) \subseteq \operatorname{cl}(P \cap \nabla P) \subseteq \overline{H}^+$.
- Hence $Q \cap \nabla Q \subseteq H^- \cap \overline{H}^+ = \varnothing$, a contradiction.

Where we are in the plan...

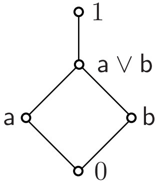
- Given a countable, completely normal distributive lattice D with zero, construct inductively a closed, surjective lattice homomorphism $f = \bigcup_{n<\omega} f_n \colon \mathrm{Id_c}\,\mathrm{F}_\ell(\omega) \twoheadrightarrow D$, where (using Baker-Beynon duality) all $E_n = \mathrm{Op}^-(\mathcal{H}_n) \underset{\mathrm{def}}{=} \mathrm{Op}(\mathcal{H}_n) \setminus \{\mathbb{R}^{(\omega)}\}$ and $f_n \colon E_n \to D$.

Where we are in the plan...

- Given a countable, completely normal distributive lattice D with zero, construct inductively a closed, surjective lattice homomorphism $f = \bigcup_{n<\omega} f_n \colon \mathrm{Id_c}\, \mathrm{F}_\ell(\omega) \twoheadrightarrow D$, where (using Baker-Beynon duality) all $E_n = \mathrm{Op}^-(\mathcal{H}_n) \underset{\mathrm{def}}{=} \mathrm{Op}(\mathcal{H}_n) \setminus \{\mathbb{R}^{(\omega)}\}$ and $f_n \colon E_n \to D$.
- The Extension Lemma for $\mathrm{Op}(\mathcal{H})$ makes it possible to ensure $\mathrm{Id_c}\, \mathrm{F}_\ell(\omega) = \bigcup_{n<\omega} E_n$ (i.e., f defined everywhere).

Where we are in the plan. . .

- Given a countable, completely normal distributive lattice D with zero, construct inductively a closed, surjective lattice homomorphism $f = \bigcup_{n<\omega} f_n \colon \mathrm{Id_c}\, \mathrm{F}_\ell(\omega) \twoheadrightarrow D$, where (using Baker-Beynon duality) all $E_n = \mathrm{Op}^-(\mathcal{H}_n) \underset{\mathrm{def}}{=} \mathrm{Op}(\mathcal{H}_n) \setminus \{\mathbb{R}^{(\omega)}\}$ and $f_n \colon E_n \to D$.
- The Extension Lemma for $\mathrm{Op}(\mathcal{H})$ makes it possible to ensure $\mathrm{Id_c}\, \mathrm{F}_\ell(\omega) = \bigcup_{n<\omega} E_n$ (i.e., f defined everywhere).
- (Ensuring f surjective) If H is "independent" from $\mathcal{H}$, then $\mathrm{Op}(\mathcal{H} \cup \{H\}) \cong \mathrm{Op}(\mathcal{H}) * \mathrm{J}_2$ (free distributive product), where J_2 is



. . . and what remains to be done

- We want to ensure f be closed!

... and what remains to be done

- We want to ensure f be closed! (i.e., $f(a) \leq f(b) \vee c \Rightarrow (\exists x)$ $a \leq b \vee x$ and $f(x) \leq c$)

...and what remains to be done

- We want to ensure f be closed! (i.e., $f(a) \leq f(b) \vee c \Rightarrow (\exists x)$ $a \leq b \vee x$ and $f(x) \leq c$)
- Given $f_n \colon \mathrm{Op}^-(\mathcal{H}_n) \to D$, $U, V \in \mathrm{Op}^-(\mathcal{H}_n)$, and $\gamma \in L$ such that $f_n(U) \leq f_n(V) \vee \gamma$, we want to find $\mathcal{H}_{n+1}$, $X \in \mathrm{Op}^-(\mathcal{H}_{n+1})$, and f_{n+1} such that $U \subseteq V \cup X$ and $f_{n+1}(X) \leq \gamma$.

. . . and what remains to be done

- We want to ensure f be closed! (i.e., $f(a) \leq f(b) \vee c \Rightarrow (\exists x)$ $a \leq b \vee x$ and $f(x) \leq c$)
- Given $f_n \colon \mathrm{Op}^{-}(\mathcal{H}_n) \to D$, $U, V \in \mathrm{Op}^{-}(\mathcal{H}_n)$, and $\gamma \in L$ such that $f_n(U) \leq f_n(V) \vee \gamma$, we want to find $\mathcal{H}_{n+1}$, $X \in \mathrm{Op}^{-}(\mathcal{H}_{n+1})$, and f_{n+1} such that $U \subseteq V \cup X$ and $f_{n+1}(X) \leq \gamma$.
- By the earlier lemmas about consonance (and some amount of work), it is sufficient to do this in case $U = A^{+}$ and $V = B^{+}$, where $A, B \in \mathcal{H}_n$.

. . . and what remains to be done

- We want to ensure f be closed! (i.e., $f(a) \leq f(b) \vee c \Rightarrow (\exists x)$ $a \leq b \vee x$ and $f(x) \leq c$)
- Given $f_n \colon \mathrm{Op}^-(\mathcal{H}_n) \to D$, $U, V \in \mathrm{Op}^-(\mathcal{H}_n)$, and $\gamma \in L$ such that $f_n(U) \leq f_n(V) \vee \gamma$, we want to find $\mathcal{H}_{n+1}$, $X \in \mathrm{Op}^-(\mathcal{H}_{n+1})$, and f_{n+1} such that $U \subseteq V \cup X$ and $f_{n+1}(X) \leq \gamma$.
- By the earlier lemmas about consonance (and some amount of work), it is sufficient to do this in case $U = A^+$ and $V = B^+$, where $A, B \in \mathcal{H}_n$.
- "Correct any instance of $f(A^+) \leq f(B^+) \vee \gamma$".

Forcing closedness of a consonant homomorphism

Let $\mathbb{E} := \mathbb{R}^{(\omega)}$, with canonical inner product $(x|y) \underset{\text{def}}{=} \sum_{n<\omega} x_n y_n$ and weak topology (making all $(x|\_)$ continuous).

Forcing closedness of a consonant homomorphism

Let $\mathbb{E} := \mathbb{R}^{(\omega)}$, with canonical inner product $(x|y) \underset{\text{def}}{=} \sum_{n<\omega} x_n y_n$ and weak topology (making all $(x|\_)$ continuous).

Lemma

Let $\mathcal{H}$ be a finite set of closed hyperplanes, let $A = \ker(a)$ and $B = \ker(b)$ in $\mathcal{H}$. Set $C_m \underset{\text{def}}{=} \ker(a - mb)$ and $\mathcal{H}_m \underset{\text{def}}{=} \mathcal{H} \cup \{C_m\}$, $\forall m \in \mathbb{N}$.

Forcing closedness of a consonant homomorphism

Let $\mathbb{E} := \mathbb{R}^{(\omega)}$, with canonical inner product $(x|y) \underset{\mathrm{def}}{=} \sum_{n<\omega} x_n y_n$ and weak topology (making all $(x|\_)$ continuous).

Lemma

Let $\mathcal{H}$ be a finite set of closed hyperplanes, let $A = \ker(a)$ and $B = \ker(b)$ in $\mathcal{H}$. Set $C_m \underset{\mathrm{def}}{=} \ker(a - mb)$ and $\mathcal{H}_m \underset{\mathrm{def}}{=} \mathcal{H} \cup \{C_m\}$, $\forall m \in \mathbb{N}$. Let L be a finite distributive lattice and let $f\colon \mathrm{Op}(\mathcal{H}) \to L$ be a consonant homomorphism.

Forcing closedness of a consonant homomorphism

Let $\mathbb{E} := \mathbb{R}^{(\omega)}$, with canonical inner product $(x|y) \underset{\mathrm{def}}{=} \sum_{n<\omega} x_n y_n$ and weak topology (making all $(x|\_)$ continuous).

Lemma

Let $\mathcal{H}$ be a finite set of closed hyperplanes, let $A = \ker(a)$ and $B = \ker(b)$ in $\mathcal{H}$. Set $C_m \underset{\mathrm{def}}{=} \ker(a - mb)$ and $\mathcal{H}_m \underset{\mathrm{def}}{=} \mathcal{H} \cup \{C_m\}$, $\forall m \in \mathbb{N}$. Let L be a finite distributive lattice and let $f \colon \mathrm{Op}(\mathcal{H}) \to L$ be a consonant homomorphism. Then for all large enough m (*independent of* L), f extends to a homomorphism $g \colon \mathrm{Op}(\mathcal{H}_m) \to L$ such that $g(A^+ \smallsetminus_{\mathrm{Op}(\mathcal{H}_m)} B^+) = f(A^+) \smallsetminus_L f(B^+)$.

Forcing closedness of a consonant homomorphism

Let $\mathbb{E} := \mathbb{R}^{(\omega)}$, with canonical inner product $(x|y) \underset{\text{def}}{=} \sum_{n<\omega} x_n y_n$ and weak topology (making all $(x|\_)$ continuous).

Lemma

Let $\mathcal{H}$ be a finite set of closed hyperplanes, let $A = \ker(a)$ and $B = \ker(b)$ in $\mathcal{H}$. Set $C_m \underset{\text{def}}{=} \ker(a - mb)$ and $\mathcal{H}_m \underset{\text{def}}{=} \mathcal{H} \cup \{C_m\}$, $\forall m \in \mathbb{N}$. Let L be a finite distributive lattice and let $f\colon \mathrm{Op}(\mathcal{H}) \to L$ be a consonant homomorphism. Then for all large enough m (*independent of* L), f extends to a homomorphism $g\colon \mathrm{Op}(\mathcal{H}_m) \to L$ such that $g(A^+ \smallsetminus_{\mathrm{Op}(\mathcal{H}_m)} B^+) = f(A^+) \smallsetminus_L f(B^+)$.

- "Large enough": setting $C_m^- \underset{\text{def}}{=} \{x \mid a(x) < mb(x)\}$ and $B^+ \underset{\text{def}}{=} \{x \mid b(x) > 0\}$, we need $\forall X \in \mathrm{Op}(\mathcal{H})$, $C_m^- \subseteq X \Rightarrow B^+ \subseteq X$.

Forcing closedness of a consonant homomorphism

Let $\mathbb{E} := \mathbb{R}^{(\omega)}$, with canonical inner product $(x|y) \underset{\mathrm{def}}{=} \sum_{n<\omega} x_n y_n$ and weak topology (making all $(x|\_)$ continuous).

Lemma

Let $\mathcal{H}$ be a finite set of closed hyperplanes, let $A = \ker(a)$ and $B = \ker(b)$ in $\mathcal{H}$. Set $C_m \underset{\mathrm{def}}{=} \ker(a - mb)$ and $\mathcal{H}_m \underset{\mathrm{def}}{=} \mathcal{H} \cup \{C_m\}$, $\forall m \in \mathbb{N}$. Let L be a finite distributive lattice and let $f\colon \mathrm{Op}(\mathcal{H}) \to L$ be a consonant homomorphism. Then for all large enough m (*independent of* L), f extends to a homomorphism $g\colon \mathrm{Op}(\mathcal{H}_m) \to L$ such that $g(A^+ \smallsetminus_{\mathrm{Op}(\mathcal{H}_m)} B^+) = f(A^+) \smallsetminus_L f(B^+)$.

- "Large enough": setting $C_m^- \underset{\mathrm{def}}{=} \{x \mid a(x) < mb(x)\}$ and $B^+ \underset{\mathrm{def}}{=} \{x \mid b(x) > 0\}$, we need $\forall X \in \mathrm{Op}(\mathcal{H})$, $C_m^- \subseteq X \Rightarrow B^+ \subseteq X$.
- Existence of m ensured by Farkas' Lemma (Hahn-Banach Theorem).

Conclusion

Putting all this together (with some work), the proof can be concluded.

Conclusion

Putting all this together (with some work), the proof can be concluded.

Corollary

For any countable ℓ-group G, there exists a countable Abelian ℓ-group A such that the lattices of all convex ℓ-subgroups of G and A are isomorphic.

Conclusion

Putting all this together (with some work), the proof can be concluded.

Corollary

For any countable ℓ-group G, there exists a countable Abelian ℓ-group A such that the lattices of all convex ℓ-subgroups of G and A are isomorphic.

- Uncountable analogue of corollary above: fails (Kenoyer 1984, McCleary 1986).

Conclusion

Putting all this together (with some work), the proof can be concluded.

Corollary

For any countable ℓ-group G, there exists a countable Abelian ℓ-group A such that the lattices of all convex ℓ-subgroups of G and A are isomorphic.

- Uncountable analogue of corollary above: fails (Kenoyer 1984, McCleary 1986).
- About real spectra now.

Conclusion

Putting all this together (with some work), the proof can be concluded.

Corollary

For any countable ℓ-group G, there exists a countable Abelian ℓ-group A such that the lattices of all convex ℓ-subgroups of G and A are isomorphic.

- Uncountable analogue of corollary above: fails (Kenoyer 1984, McCleary 1986).
- About real spectra now. The real spectrum of any commutative, unital ring is known to be a completely normal spectral space.

Conclusion

Putting all this together (with some work), the proof can be concluded.

Corollary

For any countable ℓ-group G, there exists a countable Abelian ℓ-group A such that the lattices of all convex ℓ-subgroups of G and A are isomorphic.

- Uncountable analogue of corollary above: fails (Kenoyer 1984, McCleary 1986).
- About real spectra now. The real spectrum of any commutative, unital ring is known to be a completely normal spectral space.

Corollary

For every countable commutative unital ring R, there exists a countable Abelian ℓ-group G with unit such that $\mathrm{Spec}_\ell\, G$ is homeomorphic to the real spectrum of R.