Projective classes as images of accessible functors

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May 2022
References


Motivation

We would like to prove that certain “naturally defined” categories $\mathcal{C}$ of models (say of first-order theories) are “intractable”.

Examples:
- Posets of finitely generated ideals of rings,
- Ordered $K_0$ groups of unit-regular rings,
- Stone duals of spectra of abelian lattice-ordered groups, ...
- and many other classes.

A way to define intractability is to state that $\mathcal{C}$ is not the class of models of any infinitary (not just first-order!) sentence (we’ll say elementary).

We will use a stronger notion of intractability.
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Introducing a motivating example

■ For an Abelian \( \ell \)-group \( G \), \( \text{Id}_c \ G \stackrel{\text{def}}{=} (\text{lattice of all principal } \ell \text{-ideals of } G) = \{ \langle a \rangle \mid a \in G^+ \} \) where \( \langle a \rangle \stackrel{\text{def}}{=} \{ x \in G \mid (\exists n < \omega)(|x| \leq na) \} \). Let \( \text{Id}_c A \stackrel{\text{def}}{=} \{ D \mid (\exists G)(D \cong \text{Id}_c \ G) \} \).
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- Every member of $\text{Id}_c A$ is a distributive 0-lattice. It is completely normal (abbrev. CN), that is, it satisfies

$$(\forall a, b)(\exists x, y)(a \lor b = a \lor y = x \lor b \& x \land y = 0).$$
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Every member of $\text{Id}_c \mathcal{A}$ has countably based differences (abbrev. CBD), that is, it satisfies

$$(\forall a, b)(\exists_{n < \omega} c_n)(\forall x)(a \leq b \lor x \Leftrightarrow c_n \leq x \text{ for some } n).$$
Motivating example (cont’d): Ploščica’s Condition

- For an ideal $I$ in a distributive lattice $D$, $x \equiv_I y$ if $(\exists z \in I)(x \lor z = y \lor z)$. Set $D/I \overset{\text{def}}{=} D/\equiv_I$. 

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- Tuuri’s Interpolation Theorem
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- A bounded distributive lattice $D$ satisfies Ploščica’s Condition (abbrev. Plo) if for every $a \in D$ and every collection $(m_i \mid i \in I)$ of maximal ideals of $\downarrow a$, $\downarrow a/\bigcap_i m_i$ has cardinality $\leq 2^{\text{card } I}$.
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Theorem (Ploščica 2021)

Every member of $\text{Id}_c \mathcal{A}$ satisfies Plo. On the other hand, $0\text{-DLat} & \text{CN} & \text{CBD}$ does not imply Plo.
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Every member of \( \text{Id}_c \mathcal{A} \) satisfies Plo. On the other hand, \( 0\text{-DLat}\&\text{CN}\&\text{CBD} \) does not imply Plo.

**Question:** Does the conjunction \( 0\text{-DLat}\&\text{CN}\&\text{CBD}\&\text{Plo} \) (and more...) characterize the members of \( \text{Id}_c \mathcal{A} \)?
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Every member of $\text{Id}_c \mathcal{A}$ satisfies Plo. On the other hand, $0\text{-DLat}&\text{CN}&\text{CBD}$ does not imply Plo.

- **Question**: Does the conjunction $0\text{-DLat}&\text{CN}&\text{CBD}&\text{Plo}$ (and more...) characterize the members of $\text{Id}_c \mathcal{A}$?

- **Answer**: A strong NO under (a fragment of) GCH, with a counterexample of cardinality $\aleph_4$. 
**v-structures**

- **Vocabulary:** $v = (v_{\text{ope}}, v_{\text{rel}}, \text{ar})$ with $v_{\text{ope}} \cap v_{\text{rel}} = \emptyset$ and $\text{ar} : v_{\text{ope}} \cup v_{\text{rel}} \to \text{ordinals}$ (usually) with $0 \notin \text{ar}[v_{\text{rel}}]$. 

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**Projective classes as images of accessible functors**

Terms: closure of variables under all function symbols.

Atomic formulas: $s = t$, for terms $s$ and $t$, or $R(t_{\xi} | \xi \in \text{ar}(R))$ where the $t_{\xi}$ are terms and $R \in v_{\text{rel}}$. 

$\text{Str}(v)$ def = category of all $v$-structures with $v$-homomorphisms (it is locally presentable).
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- **\( \text{Str}(v) \overset{\text{def}}{=} \)** category of all \( v \)-structures with \( v \)-homomorphisms (it is **locally presentable**).
- **Terms**: closure of variables under all functions symbols.
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The languages $\mathcal{L}_{\kappa\lambda}$

- Here $\kappa$ and $\lambda$ are “extended cardinals” ($\infty$ allowed) with $\omega \leq \lambda \leq \kappa \leq \infty$. 

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- Satisfaction $A \models E(\vec{a})$ defined as usual ($A$ is a $\mathbf{v}$-structure, $E \in \mathcal{L}_{\infty\infty}(\mathbf{v})$, $\vec{a}$: free variables ($E \to A$).
The languages $L_{\kappa \lambda}$

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- Satisfaction $A \models E(\bar{a})$ defined as usual ($A$ is a $v$-structure, $E \in L_{\infty \infty}(v)$, $\bar{a}$: free variables ($E \to A$).

- $L_{\kappa \lambda}$-elementary class:
  $C = \text{Mod}_v(E) \overset{\text{def}}{=} \{ A \in \text{Str}(v) \mid A \models E \}$ where $E$ is an $L_{\kappa \lambda}(v)$-sentence.
A class $\mathcal{C}$ of $v$-structures is

\[ \text{(Relatively) projective classes} \]

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\[ \text{Hence } \text{PC}(L_{\kappa\lambda}) \subseteq \text{RPC}(L_{\kappa\lambda}). \text{ Note that } \text{PC}(L_{\omega\omega}) \nsubseteq \text{RPC}(L_{\omega\omega}) \text{ (even on finite structures).} \]

\[ \text{Theorem (W 2021)} \]

\[ \text{Let } \lambda \text{ be an infinite cardinal. Then } \text{PC}(L_{\infty\lambda}) = \text{RPC}(L_{\infty\lambda}) \text{ (in full generality; no restrictions on vocabularies). Moreover, if } \lambda \text{ is singular, then } \text{PC}(L_{\infty\lambda}) = \text{PC}(L_{\infty\lambda^+}). \]
(Relatively) projective classes

A class \( \mathcal{C} \) of \( \nu \)-structures is

- **projective over** \( \mathcal{L}_{\kappa, \lambda} \) (abbrev. \( \text{PC}(\mathcal{L}_{\kappa, \lambda}) \)) if there are a vocabulary \( \nu \supseteq \mathcal{V} \) and a sentence \( E \in \mathcal{L}_{\kappa, \lambda}(\mathcal{W}) \) such that
  \[
  \mathcal{C} = \{ \mathcal{M} \upharpoonright \mathcal{V} \mid \mathcal{M} \in \text{Mod}_{\mathcal{W}}(E) \}.
  \]

- **relatively projective over** \( \mathcal{L}_{\kappa, \lambda} \) (abbrev. \( \text{RPC}(\mathcal{L}_{\kappa, \lambda}) \)) if there are a unary predicate symbol \( U \), a vocabulary \( \nu \supseteq \mathcal{V} \cup \{ U \} \), and a sentence \( E \in \mathcal{L}_{\kappa, \lambda}(\mathcal{W}) \) such that
  \[
  \mathcal{C} = \{ U^M \upharpoonright \mathcal{V} \mid M \in \text{Mod}_{\mathcal{W}}(E), \ U^M \text{ closed under } \nu_{\text{ope}} \}.
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(Relatively) projective classes

A class $\mathcal{C}$ of $\nu$-structures is

- **projective over** $\mathcal{L}_{\kappa\lambda}$ (abbrev. $PC(\mathcal{L}_{\kappa\lambda})$) if there are a vocabulary $\omega \supseteq \nu$ and a sentence $E \in \mathcal{L}_{\kappa\lambda}(\omega)$ such that
  $$\mathcal{C} = \{ M \upharpoonright\nu \mid M \in \text{Mod}_\omega(E) \}.$$  

- **relatively projective over** $\mathcal{L}_{\kappa\lambda}$ (abbrev. $\text{RPC}(\mathcal{L}_{\kappa\lambda})$) if there are a unary predicate symbol $U$, a vocabulary $\omega \supseteq \nu \cup \{U\}$, and a sentence $E \in \mathcal{L}_{\kappa\lambda}(\omega)$ such that
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- Hence $PC(\mathcal{L}_{\kappa\lambda}) \subseteq \text{RPC}(\mathcal{L}_{\kappa\lambda})$. Note that $PC(\mathcal{L}_{\omega\omega}) \subsetneq \text{RPC}(\mathcal{L}_{\omega\omega})$ (even on finite structures).
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A class \( \mathcal{C} \) of \( \mathbf{v} \)-structures is

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- Hence \( \text{PC}(\mathcal{L}_{\kappa\lambda}) \subseteq \text{RPC}(\mathcal{L}_{\kappa\lambda}) \). Note that \( \text{PC}(\mathcal{L}_{\omega\omega}) \not\subseteq \text{RPC}(\mathcal{L}_{\omega\omega}) \) (even on finite structures).

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Let \( \lambda \) be an infinite cardinal. Then \( \text{PC}(\mathcal{L}_{\infty\lambda}) = \text{RPC}(\mathcal{L}_{\infty\lambda}) \) (in full generality; no restrictions on vocabularies). Moreover, if \( \lambda \) is singular, then \( \text{PC}(\mathcal{L}_{\infty\lambda}) = \text{PC}(\mathcal{L}_{\infty\lambda^+}) \).
Examples of “elementary” classes

- **Finiteness** (of the amiant universe) is $L_{\omega_1 \omega}$:
  \[
  \bigwedge_{n<\omega} (\exists i<n x_i) (\forall x) \bigwedge_{i<n} (x = x_i).
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- **Well-foundedness** (of the ambiant poset) is $\mathcal{L}_{\omega_1 \omega_1}$:
  \[
  (\forall_{n<\omega} x_n) \bigwedge_{n<\omega} (x_{n+1} \not\prec x_n).
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- **Finiteness** (of the ambiant universe) is $\mathcal{L}_{\omega_1\omega}$:

  $$\forall n<\omega \left( \exists i<n x_i \right)(\forall x) \bigwedge_{i<n} (x = x_i).$$

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  $$\left( \forall n<\omega x_n \right) \bigwedge_{n<\omega} (x_{n+1} \nless x_n).$$

- **Torsion-freeness** (of a group) is $\mathcal{L}_{\omega_1\omega}$:

  $$\forall 0<n<\omega (\forall x) (x^n = 1 \implies x = 1).$$
An example of RPC (that turns out to be PC)

\[ \mathcal{C} \overset{\text{def}}{=} \{ M = (M, \cdot, 1) \text{ monoid} \mid (\exists G \text{ group})(M \hookrightarrow G) \} \]

is, by definition, RPC(\(L_{\omega\omega}\)).
An example of RPC (that turns out to be PC)

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- Here \( v = (\cdot, 1) \), \( w = (\cdot, 1, U) \) for a unary predicate \( U \), the required \( E \) states that the given \( w \)-structure is a group (so “\( U^G \) is \( v \)-closed in \( G \)” means that \( U \) interprets a submonoid of \( G \)).
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- By Mal’cev’s work, \( \mathcal{C} = \{ M \mid (\forall n < \omega)(M \models E_n) \} \) for an effectively constructed sequence \((E_n \mid n < \omega)\) of quasi-identities over \( v \), not reducible to any finite subset.
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- Nonetheless, \( \mathcal{C} = \{ M \mid (\exists \text{ group structure } G \text{ on } M)(\exists f : M \hookrightarrow G) \} \) is PC(\( \mathcal{L}_{\omega\omega} \)).
Other examples

- For a unital ring $R$, $\text{Id}_c R \overset{\text{def}}{=} (\lor, 0)$-semilattice of all finitely generated two-sided ideals of $R$. Let $\mathcal{C} \overset{\text{def}}{=} \{ \text{Id}_c R \mid R \text{ unital ring} \}$ (up to isomorphism).
Other examples

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- For a commutative unital ring $A$, $\Phi(A) \overset{\text{def}}{=} \text{Stone dual of the real spectrum of } A$ (it is a bounded distributive lattice). Let $\mathcal{C} \overset{\text{def}}{=} \{ \Phi(A) \mid A \text{ commutative unital ring}\}$. 

All those classes are $\mathcal{P}(\mathcal{L}_{\omega_1 \omega})$. Observe that they are all defined as images of functors. We will see that none of those classes is $\text{co-}\mathcal{P}(\mathcal{L}_{\infty \infty})$ (i.e., complement of a $\mathcal{P}(\mathcal{L}_{\infty \infty})$).
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- A functor $\Phi: \mathcal{S} \to \mathcal{T}$ is $\lambda$-continuous if it preserves $\lambda$-directed colimits. If $\mathcal{S}$ and $\mathcal{T}$ are both $\lambda$-accessible categories, we say that $\Phi$ is a $\lambda$-accessible functor.
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- There are many examples: \( \text{Str}(v) \), quasivarieties. . .
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**Theorem (W 2021)**

Let $\lambda$ be a regular cardinal, let $v$ be a $\lambda$-ary vocabulary, and let $C$ be a class of $v$-structures. Then TFAE:

1. $C$ is PC($L_{\infty \lambda}$)- (resp., RPC($L_{\infty \lambda}$))-definable.

2. There are a $\lambda$-accessible category $S$ and a $\lambda$-continuous functor (that can then be taken faithful) $\Phi: S \to \text{Str}(v)$ with $\Phi(S) = C$. 

Recall that $\Phi(S) \overset{\text{def}}{=} \{ M \mid (\exists S \in \text{Ob}S) (M \cong \Phi(S)) \}$. 

The assumption that $v$ be $\lambda$-ary cannot be dispensed with (counterexamples for both directions, involving idempotence and emptiness, respectively).
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Infinitely deep languages

- **Idea**: extend $L_{\kappa\lambda}$ in such a way that infinite alternations of quantifiers be enabled.
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- **Game formula** (of Gale-Stewart kind): $\exists \vec{x} E(\vec{x})$ is $(\forall x_0)(\exists x_1)(\forall x_2) \cdots E(x_0, x_1, x_2, \ldots)$. 
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- Can be interpreted via a game with two players, $\forall$ (who plays all $x_{2n}$) and $\exists$ (who plays all $x_{2n+1}$). Hence $\forall$ (resp., $\exists$) wins iff $E(x_0, x_1, x_2, \ldots)$ (resp., $\neg E(x_0, x_1, x_2, \ldots)$).
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- The “infinitely deep language” $\mathcal{M}_{\kappa\lambda}(\nu)$ contains more general formulas than the $\exists \vec{x} \, E(\vec{x})$ above, now clocked by posets which are simultaneously trees and meet-semilattices, in which every node has $< \kappa$ upper covers and every branch has length a successor $< \lambda$. 
Infinitely deep languages

- **Idea**: extend $\mathcal{L}_{\kappa,\lambda}$ in such a way that infinite alternations of quantifiers be enabled.
- **Game formula** (of Gale-Stewart kind): $\exists\vec{x} E(\vec{x})$ is $(\forall x_0)(\exists x_1)(\forall x_2) \cdots E(x_0, x_1, x_2, \ldots)$.
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- The “infinitely deep language” $\mathcal{M}_{\kappa,\lambda}(\mathbf{v})$ contains more general formulas than the $\exists\vec{x} E(\vec{x})$ above, now clocked by posets which are simultaneously trees and meet-semilattices, in which every node has $<\kappa$ upper covers and every branch has length a successor $<\lambda$.
- **Satisfaction** of an $\mathcal{M}_{\kappa,\lambda}(\mathbf{v})$-statement is expressed via the existence of a winning strategy in the associated game.
Tuuri’s Interpolation Theorem

**Theorem (Tuuri 1992)**

Let $\kappa$ be a regular cardinal, let $v$ be a $\kappa$-ary vocabulary, set

$$\lambda \overset{\text{def}}{=} \sup\{\kappa^\alpha \mid \alpha < \kappa\},$$

and let $E$ and $F$ be $L_{\kappa^+\kappa}(v)$-sentences such that the conjunction $E \land F$ has no $v$-model. Then there exists an $M_{\lambda^+\lambda}(v)$-sentence $G$, with vocabulary the intersection of the vocabularies of $E$ and $F$, such that $\models (E \Rightarrow G)$ and $\models (G \Rightarrow \neg F)$.

Here, $\neg G$ denotes the sentence obtained by interchanging $\lor \lor$ and $\land \land$, $\exists$ and $\forall$, $A$ and $\neg A$ in the expression of $G$ by a tree-clocked game; it implies the usual negation $\neg G$ (which, however, is no longer an $M_{\lambda^+\lambda}$-sentence).

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Let $\kappa$ be a regular cardinal, let $\mathbf{v}$ be a $\kappa$-ary vocabulary, set

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and let $E$ and $F$ be $L_{\kappa+\kappa}(\mathbf{v})$-sentences such that the conjunction $E \land F$ has no $\mathbf{v}$-model. Then there exists an $M_{\lambda+\lambda}(\mathbf{v})$-sentence $G$, with vocabulary the intersection of the vocabularies of $E$ and $F$, such that $|\models (E \Rightarrow G)$ and $|\models (G \Rightarrow \neg F)$.

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- By a 1971 counterexample due to Malitz, $M_{\lambda+\lambda}$ cannot be replaced by $L_{\infty\infty}$ in the statement of Tuuri’s Theorem.
Corollary

Let \( v \) be a vocabulary. Then for all classes \( \mathcal{A} \) and \( \mathcal{B} \) of \( v \)-structures, if \( \mathcal{A} \) is PC(\( L_{\infty\infty} \)), \( \mathcal{B} \) is co-PC(\( L_{\infty\infty} \)), and \( \mathcal{A} \subseteq \mathcal{B} \), then there exists an \( M_{\infty\infty}(v) \)-sentence \( G \) such that \( \mathcal{A} \subseteq \text{Mod}_v(G) \subseteq \mathcal{B} \).
Projective and co-projective

**Corollary**

Let $\mathbf{v}$ be a vocabulary. Then for all classes $\mathcal{A}$ and $\mathcal{B}$ of $\mathbf{v}$-structures, if $\mathcal{A}$ is $\text{PC}(L_{\infty\infty})$, $\mathcal{B}$ is $\text{co-PC}(L_{\infty\infty})$, and $\mathcal{A} \subseteq \mathcal{B}$, then there exists an $M_{\infty\infty}(\mathbf{v})$-sentence $G$ such that $\mathcal{A} \subseteq \text{Mod}_\mathbf{v}(G) \subseteq \mathcal{B}$.

**Corollary**

In order to prove that a $\text{PC}(L_{\infty\infty})$ class $\mathcal{C}$ of $\mathbf{v}$-structures is not $\text{co-PC}(L_{\infty\infty})$, it suffices to prove that $\mathcal{C}$ is not $M_{\infty\infty}(\mathbf{v})$-definable.
Corollary

Let $\mathbf{v}$ be a vocabulary. Then for all classes $\mathcal{A}$ and $\mathcal{B}$ of $\mathbf{v}$-structures, if $\mathcal{A}$ is $\text{PC}(\mathcal{L}_{\infty\infty})$, $\mathcal{B}$ is $\text{co-PC}(\mathcal{L}_{\infty\infty})$, and $\mathcal{A} \subseteq \mathcal{B}$, then there exists an $\mathcal{M}_{\infty\infty}(\mathbf{v})$-sentence $G$ such that $\mathcal{A} \subseteq \text{Mod}_\mathbf{v}(G) \subseteq \mathcal{B}$.

Corollary

In order to prove that a $\text{PC}(\mathcal{L}_{\infty\infty})$ class $\mathcal{C}$ of $\mathbf{v}$-structures is not $\text{co-PC}(\mathcal{L}_{\infty\infty})$, it suffices to prove that $\mathcal{C}$ is not $\mathcal{M}_{\infty\infty}(\mathbf{v})$-definable.

But then, what is the advantage of $\mathcal{M}_{\infty\infty}$-definable over $\text{PC}(\mathcal{L}_{\infty\infty})$-definable or $\text{co-PC}(\mathcal{L}_{\infty\infty})$-definable?
There are several non-equivalent definitions of back-and-forth between models (extended to categorical model theory by Beke and Rosický in 2018).
That's back-and-forth!

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**Definition (Karttunen 1979)**

For a regular cardinal $\lambda$, a $\lambda$-back-and-forth system between models $M$ and $N$ over a vocabulary $\mathbf{v}$ consists of a poset $(\mathcal{F}, \trianglelefteq)$, together with a function $f \mapsto \overline{f}$ with domain $\mathcal{F}$, such that each $\overline{f} : d(f) \overset{\sim}{\rightarrow} r(f)$ with $d(f) \leq M$ and $r(f) \leq N$, and the following conditions hold:

1. $f \trianglelefteq g$ implies $\overline{f} \subseteq \overline{g}$;
2. $(\mathcal{F}, \trianglelefteq)$ is $\lambda$-inductive;
3. whenever $f \in \mathcal{F}$ and $x \in M$ (resp., $y \in N$), there is $g \in \mathcal{F}$ such that $f \subseteq g$ and $x \in d(g)$ (resp., $y \in r(g)$).

We then write $M \leftrightarrow_{\lambda} N$. 
Theorem (Karttunen 1979)

Let $\lambda$ be a regular cardinal and let $M$ and $N$ be structures over a vocabulary $v$. If $M \leftrightarrow_\lambda N$, then $M$ and $N$ satisfy the same $M_{\infty \lambda}(v)$-sentences.
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- Extended by Karttunen to the even more general languages $N_{\infty\lambda}$.
- The syntax for $N_{\infty\lambda}$ is far more complex than for $M_{\infty\lambda}$, the semantics are even trickier (not unique!).
Establishing intractability

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**Proposition**

In order to prove that a \( \text{PC}(\mathcal{L}_{\infty\infty}) \) class \( \mathcal{C} \) of \( \mathbf{v} \)-structures is not \( \text{co-PC}(\mathcal{L}_{\infty\infty}) \), it suffices to prove that it is not closed under \( \Leftrightarrow_{\lambda} \) for a suitable regular cardinal \( \lambda \).
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**Proposition**

In order to prove that a \( \text{PC}(\mathcal{L}_{\infty\infty}) \) class \( \mathcal{C} \) of \( \mathbf{v} \)-structures is not \( \text{co-PC}(\mathcal{L}_{\infty\infty}) \), it suffices to prove that it is not closed under \( \leftrightarrow_\lambda \) for a suitable regular cardinal \( \lambda \).

- Applies to earlier introduced examples \( \text{Id}_c \)(unital rings), \( \text{Id}_c \)(Abelian \( \ell \)-groups), duals of real spectra of commutative unital rings, and many others: each of those classes fails to be closed under a suitable \( \leftrightarrow_\lambda \).
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**Proposition**

In order to prove that a $PC(\mathcal{L}_{\infty\infty})$ class $\mathcal{C}$ of $v$-structures is not co-$PC(\mathcal{L}_{\infty\infty})$, it suffices to prove that it is not closed under $\leftrightarrow_\lambda$ for a suitable regular cardinal $\lambda$.

- Applies to earlier introduced examples $\text{Id}_c$(unital rings), $\text{Id}_c$(Abelian $\ell$-groups), duals of real spectra of commutative unital rings, and many others: each of those classes fails to be closed under a suitable $\leftrightarrow_\lambda$.
- The real trouble is: find a back-and-forth system $\mathcal{F}: M \leftrightarrow_\lambda N$ with $M \in \mathcal{C}$ and $N \notin \mathcal{C}$ (where $\mathcal{C}$ is the given class).
In many examples, such as $\Phi(\text{unital rings})$ and $\Phi(\text{Abelian } \ell\text{-groups})$ (where $\Phi = \text{Id}_c$), $\leftrightarrow_{\lambda}$ arises from some $\lambda$-continuous functor $\Gamma: [\kappa]^\text{inj} \to \mathcal{C}$ with $\kappa \geq \lambda$. 
Back-and-forth systems from continuous functors

In many examples, such as $\Phi(\text{unital rings})$ and $\Phi(\text{Abelian } \ell\text{-groups})$ (where $\Phi = \text{Id}_c$), $\leftrightarrow_\lambda$ arises from some $\lambda$-continuous functor $\Gamma : [\kappa]^{\text{inj}} \to C$ with $\kappa \geq \lambda$. Here, $[\kappa]^{\text{inj}}$ denotes the category of all subsets of $\kappa$ with one-to-one functions.
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It is often the case that for $X \subseteq \kappa$ with $\text{card } X < \lambda$, 

$$
\Gamma(X) = \Phi(\prod (S_{|u|} \mid u \in X^{\subseteq P})) \text{ (a “condensate”), where:}
$$

1. $P$ is a suitable finite lattice (in both examples above, $P = \{0, 1\}^3$; also, this method provably fails for arbitrary finite bounded posets!);
2. $X^{\subseteq P} \overset{\text{def}}{=} \bigcup \{X^D \mid D \subseteq P\}$;
3. $|u| \overset{\text{def}}{=} \bigvee \text{dom } u$ whenever $u \in X^{\subseteq P}$;
4. $\vec{S}$ is a non-commutative diagram, indexed by $P$, such that, for the given functor $\Phi$, the diagram $\Phi(\vec{S})$ is commutative.
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- Finding $P$ and $\vec{S}$ is usually hard, very much connected to the algebraic and combinatorial data of the given problem.
The diagram \( \tilde{S} \) for \( \text{Id}_c(\text{Abelian } \ell\text{-groups}) \)

\[
\begin{align*}
A_{123}(a, a', b, c) \\
A_{12}(a, b) & \quad A_{13}(a', c) & \quad A_{23}(b, c) \\
A_1(a) & \quad A_2(b) & \quad A_3(c) \\
A_\emptyset = \{0\}
\end{align*}
\]

\[0 \leq a \leq a' \leq 2a; \ b \geq 0; \ c \geq 0.\]
\[A_1(a) \rightarrow A_{13}(a', c) \text{ via } a \mapsto a'.\]
Thanks for your attention!