Approximating the finite by the infinite: Larders and CLL

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Most of the results discussed here obtained with **Pierre Gillibert**.

February 6, 2010

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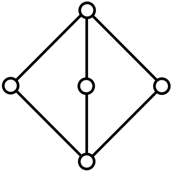

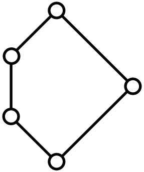

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- A lattice is modular (resp., distributive) iff it contains no copy of N_5 (resp., M_3 and N_5).

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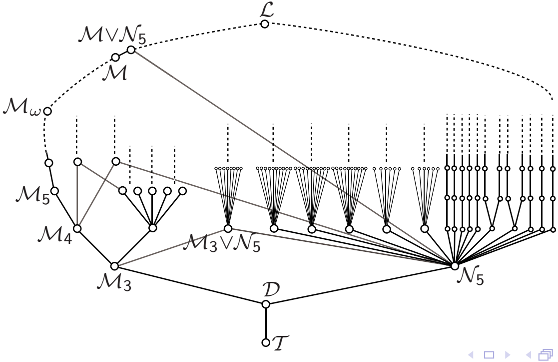

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- Finitely generated (=compact) congruence: least congruence that identifies x_1 with $y_1, \ldots, x_n$ with y_n (where $x_i, y_i \in L$ given).

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- Example: $\mathbf{crit}(\text{groups}, \text{lattices}) = 5$. On the other hand, $\mathbf{crit}(\text{lattices}, \text{groups}) = \aleph_2$ (Růžička, Tůma, and W.).

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- Lifting problems: can also be defined for more complex diagrams of finite distributive lattices and $(\vee, 0)$-homomorphisms.

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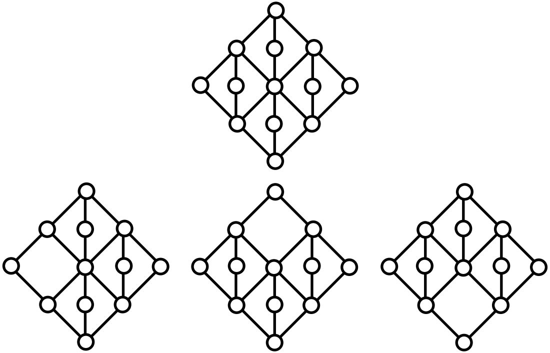

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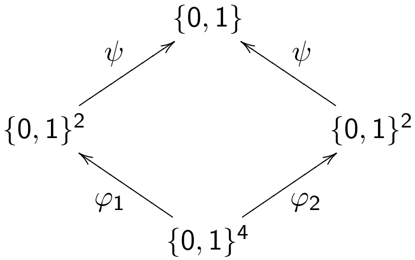

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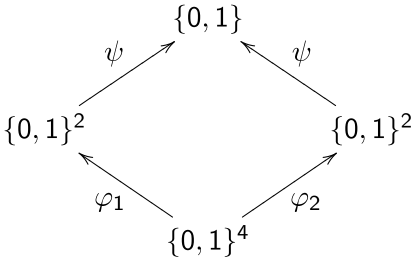

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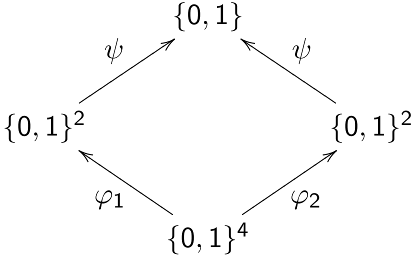

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- Prove that the diagram can be lifted in $\mathbf{Var}(A)$, but not in $\mathbf{Var}(B)$. Purely combinatorial (computational), once A, B, and the diagram have been guessed.

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- Why $\aleph_1$? This depends of the shape of the diagram (here, a square, $\{0,1\}^2$).
- The "condensation principle" above has been subsequently set into a more general, categorical, framework.

General categorical settings

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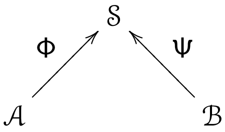

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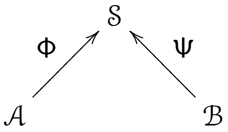

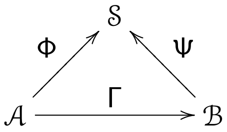

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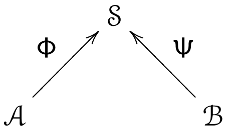

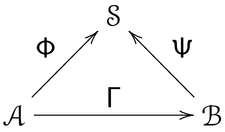

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Hence we need an assumption of the form "for many $A\in\mathcal{A}$, there exists $B\in\mathcal{B}$ such that $\Phi(A)\cong\Psi(B)$". Ask for $\Gamma\colon A\mapsto B$ to be a functor (at least on a large enough subcategory of $\mathcal{A}$).

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- For an infinite regular cardinal λ, a λ-larder consists of categories $\mathcal{A}$, $\mathcal{B}$, $\mathcal{S}$ with functors $\Phi\colon \mathcal{A} \to \mathcal{S}$ and $\Psi\colon \mathcal{B} \to \mathcal{S}$, together with a bunch of add-ons:

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- ...satisfying lots of extra properties (preservation properties related to colimits, plus an analogue of the Löwenheim-Skolem Theorem).

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In short: in order to lift the diagram $\Phi\vec{A}$ with respect to Ψ, $\Rightarrow$, it is sufficient to lift the object $\Phi(A)$ with respect to Ψ, $\Rightarrow$, where A is a suitable condensate of $\vec{A}$ (viz. $A := \mathbf{F}(X) \otimes \vec{A}$).

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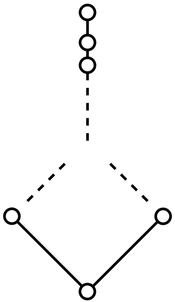
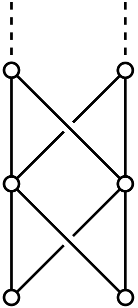
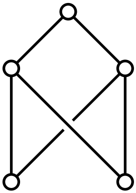

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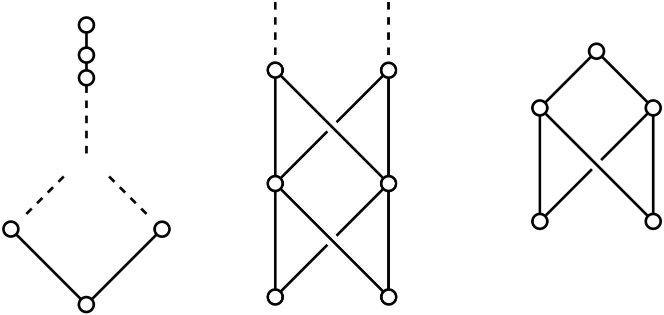

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- A lattice is coordinatizable, if it is isomorphic to $\mathbb{L}(R)$ for some regular ring R.

Non-coordinatizable 2-distributive lattices

The identity of 2-distributivity:

$$x \wedge (y_0 \vee y_1 \vee y_2) = (x \wedge (y_0 \vee y_1)) \vee (x \wedge (y_0 \vee y_2)) \vee (x \wedge (y_1 \vee y_2)) \,.$$

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- There exists a non-coordinatizable 2-distributive complemented modular lattice, of cardinality $\aleph_1$, with a spanning M_ω. In particular, coordinatizability is not first-order. (Established *via* a condensate-like construction)

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There exists a non-coordinatizable sectionally complemented modular lattice, of cardinality $\aleph_1$, with a large 4-frame.

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- Then larders are used to turn the diagram counterexample to an object counterexample.

Lattices without congruence-permutable, congruence-preserving extension

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Unlike all previous examples, the larder data for this result are difficult to figure out.