From lifting objects to lifting diagrams: recent progress on larders and CLL

Friedrich Wehrung

Université de Caen
LMNO, UMR 6139
Département de Mathématiques
14032 Caen cedex
E-mail: wehrung@math.unicaen.fr
URL: http://www.math.unicaen.fr/~wehrung
Most of the results discussed here obtained with **Pierre Gillibert***.*

August 17, 2009

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Let's see some examples.

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In fact, the functor Γ constructed in Pudlák's proof sends finite distributive lattices to finite atomistic lattices, and preserves directed colimits (=direct limits).

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No, it cannot. (For nontrivial reasons, that can be extended to any variety with a nontrivial congruence $(\vee,\wedge)$-identity.)

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Due to the link between K-theory of regular rings and congruence lattices of lattices, Růžička's result extends Schmidt's result.

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- A lattice is coordinatizable, if it is isomorphic to $\mathbb{L}(R)$ for some regular ring R.

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A spanning M_ω in a bounded lattice L is a $0, 1$-sublattice of L isomorphic to M_ω.

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$M_\omega := \{0, 1, a_0, a_1, a_2, \dots\}$, all a_i atoms, is 2-distributive. A spanning M_ω in a bounded lattice L is a $0,1$-sublattice of L isomorphic to M_ω.

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- There exists a non-coordinatizable 2-distributive complemented modular lattice, of cardinality $\aleph_1$, with a spanning M_ω. In particular, coordinatizability is not first-order.

P-normed topological spaces

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- For P-normed spaces $\mathbf{X}$ and $\mathbf{Y}$, a morphism $\mathbf{X} \to \mathbf{Y}$ is a continuous map $f \colon X \to Y$ such that $\|f(x)\|_{\mathbf{Y}} \subseteq \|x\|_{\mathbf{X}}$ for each $x \in X$.

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  A description of the dual category follows.

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where A is a Boolean algebra, each $A^{(p)}$ is an ideal of A, and

1. $A = \bigvee(A^{(p)} \mid p \in P)$ in $\operatorname{Id} A$;
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- Denote by $\mathbf{Bool}_P$ the category of all P-scaled Boolean algebras with above described morphisms.

Duality between $\mathbf{BTop}_P$ and $\mathbf{Bool}_P$

- For a P-scaled Boolean algebra $\mathbf{A}$, we set

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- The structure $\operatorname{Clop}\mathbf{X} := \big(A, (A^{(p)} \mid p \in P)\big)$ is a P-scaled Boolean algebra.

Basic categorical properties of $\mathbf{BTop}_P$ and $\mathbf{Bool}_P$

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The pair $(\mathrm{Ult}, \mathrm{Clop})$ defines a duality between the category $\mathbf{BTop}_P$ of all P-normed Boolean spaces and the category $\mathbf{Bool}_P$ of all P-scaled Boolean algebras.

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Finitely presented objects in a category

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An object A in a category $\mathcal{C}$ is finitely presented, if for every directed colimit representation

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For example, an element in a poset is finitely presented iff it is compact.

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Every P-scaled Boolean algebra is a monomorphic directed colimit of finitely presented P-scaled Boolean algebras.

Normal morphisms of P-scaled Boolean algebras

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Proposition (Gillibert and W., 2009)

Every normal morphism in $\mathbf{Bool}_P$ is a directed colimit of compact normal morphisms.

Defining $\mathbf{A} \otimes \vec{S}$ for $\mathbf{A}$ finitely presented

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- We get a $\mathcal{S}$-valued functor $\mathbf{A} \mapsto \mathbf{A} \otimes \vec{S}$, defined on the finitely presented part of $\mathbf{Bool}_P$.

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- (pseudo join-semilattice)$\Rightarrow$(supported)$\Rightarrow$(almost join-semilattice); the converses do not hold.

Norm-coverings and λ-lifters

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- Let λ be an infinite cardinal. A λ-lifter of P is a pair $(X, \mathbf{X})$, where X is a norm-covering of P, $\mathbf{X}$ is a set of sharp ideals of X, and, setting $\mathbf{X}^{=} := \{\mathbf{x} \in \mathbf{X} \mid \partial\mathbf{x} \text{ not maximal}\}$,
  1. $\operatorname{card}(\mathbf{X} \downarrow \mathbf{x}) < \lambda$ for each $\mathbf{x} \in \mathbf{X}^{=}$.
  2. (Kuratowski-like property) For each isotone $S \colon \mathbf{X}^{=} \to [X]^{<\lambda}$, there exists an isotone $\sigma \colon P \to X$ such that
     1. $\partial \circ \sigma = \mathrm{id}_P$;
     2. $\forall p < q$ in P, $S(\sigma(p)) \cap \sigma(q) \subseteq \sigma(p)$.
  3. If $\lambda = \aleph_0$, then X is supported.

The P-scaled Boolean algebras $\mathbf{F}(X)$

- For a norm-covering $\partial\colon X \to P$, denote by $\mathrm{F}(X)$ the Boolean algebra defined by generators $\tilde{u}$ (for $u \in X$) and relations
  1. $\tilde{v} \leq \tilde{u}$, for all $u \leq v$ in X;
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- The pair $\mathbf{F}(X) := \bigl(\mathrm{F}(X), (\mathrm{F}(X)^{(p)} \mid p \in P)\bigr)$ is a P-scaled Boolean algebra.
- The assignment $X \mapsto \mathbf{F}(X)$ has nice functorial properties.

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- Every finite almost join-semilattice P has a λ-lifter (λ arbitrary infinite cardinal). The minimal cardinality of a possible underlying X is $\leq \lambda^{+(\dim P - 1)}$ (and $<$ may occur).
- For infinite P, the existence of λ-lifters is related to large cardinal axioms, for instance Erdős cardinals.

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An octuple $(\mathcal{A}, \mathcal{B}, \mathcal{S}, \mathcal{A}^{\dagger}, \mathcal{B}^{\dagger}, \mathcal{S}^{\Rightarrow}, \Phi, \Psi)$ is a λ-larder, if $\mathcal{A}^{\dagger} \subseteq \mathcal{A}$ full, $\mathcal{B}^{\dagger} \subseteq \mathcal{B}$ full, $\mathcal{S}^{\Rightarrow} \subseteq \mathcal{S}$ subcategory, $B \in \mathcal{B}^{\dagger}$ is λ-presented in $\mathcal{B}$ and $\Psi(B)$ is λ-presented in $\mathcal{S}$ for each $B \in \mathcal{B}^{\dagger}$, $\mathcal{A}$ has all nonempty small directed $\varinjlim$s and all nonempty finite products, $\mathcal{S}^{\Rightarrow}$ is "closed under nonempty small directed $\varinjlim$s", Φ preserves nonempty small directed $\varinjlim$s, Ψ preserves nonempty λ-small directed $\varinjlim$s, $\Phi(\text{projections}) \subseteq \mathcal{S}^{\Rightarrow}$, and (Löwenheim-Skolem Property) for each $S \in \Phi(\mathcal{A}^{\dagger})$, each $B \in \mathcal{B}$, and each $\varphi \colon \Psi(B) \to S$ in $\mathcal{S}^{\Rightarrow}$ there are "many" $u \colon U \to B$ with $U \in \mathcal{B}^{\dagger}$ such that $\varphi \circ \Psi(u) \in \mathcal{S}^{\Rightarrow}$.

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- For example, if $\mathcal{S}$ is the category of all $(\vee, 0)$-semilattices with $(\vee, 0)$-homomorphisms, $\mathcal{S}^{\Rightarrow}$ is often the subcategory with morphisms of the form $S \twoheadrightarrow S/I$ (I ideal of S) up to iso, and then any double arrow $\varphi\colon \operatorname{Con_c} \mathbf{U} \Rightarrow S$ can be "nicely factored" through an isomorphism.

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In short: in order to lift the diagram $\Phi\vec{A}$ with respect to Ψ, $\Rightarrow$, it is sufficient to lift the object $\Phi(A)$ with respect to Ψ, $\Rightarrow$, where A is a suitable condensate of $\vec{A}$ (viz. $A := \mathbf{F}(X) \otimes \vec{A}$).

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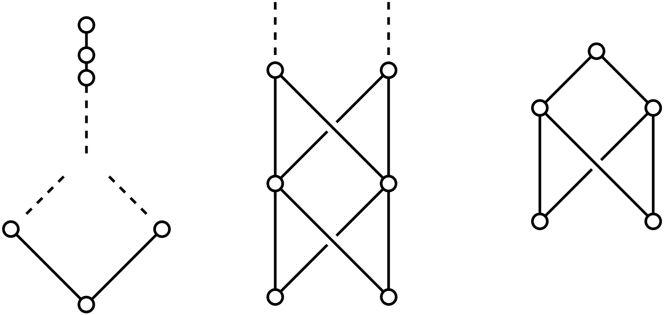

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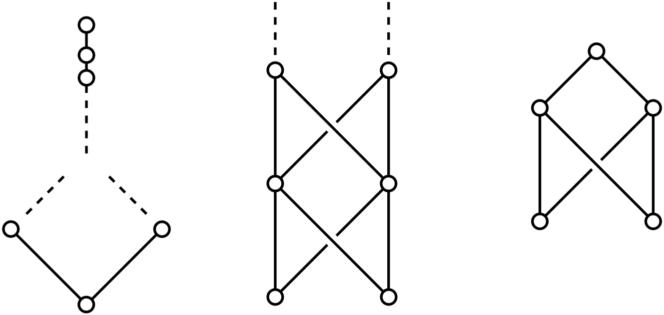

- Too bad. . .

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- Hence, if we want to state a diagram version of the GS Theorem, we need to work in a suitable category of non-indexed algebras.
- Among 3 possible definitions of non-indexed algebras, 2 of them won't satisfy the assumptions of CLL.
- The one that works is the following: consider the category $\mathbf{MAlg}_1$ of all unary algebras, where $f\colon \mathbf{A} \to \mathbf{B}$ means that $\mathrm{Op}(\mathbf{A}) \subseteq \mathrm{Op}(\mathbf{B})$ and f is a homomorphism for all symbols in $\mathrm{Op}(\mathbf{A})$.

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- For an infinite regular cardinal λ, denote by $\mathbf{Sem}_{\vee,0}^{(\lambda)}$ the class of all $(\vee,0)$-semilattices of cardinality $< \lambda$. Similarly for $\mathbf{MAlg}_1^{(\lambda)}$ (require $\operatorname{card} \mathbf{A} + \operatorname{card} \operatorname{Op}(\mathbf{A}) < \lambda$).

The Grätzer-Schmidt Theorem (picture of the larder data)

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The large cardinal axiom in question states the existence of large independent sets for certain set functions (cf. Kuratowski's Free Set Theorem). If there is a proper class of Erdős cardinals (this axiom is weaker, consistency-wise, than a Ramsey cardinal), then this assumption is satisfied for any poset P.

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- Quasivariety of structures: class of first-order structures, in a given first-order language, closed under **S**, **P**, and directed $\varinjlim$s.

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- Then set $\operatorname{Con}_{\mathrm{c,r}}\mathcal{V} := \{S \in \mathbf{Sem}_{\vee,0} \mid (\exists \mathbf{A} \in \mathcal{V})(S \cong \operatorname{Con}_{\mathrm{c}}^{\mathcal{V}}\mathbf{A})\}$.

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- For quasivarieties $\mathcal{A}$ and $\mathcal{B}$ (not necessarily in the same language), set

$$\operatorname{crit}_{\mathrm{r}}(\mathcal{A};\mathcal{B}) := \min\{\operatorname{card} S \mid S \in (\operatorname{Con}_{\mathrm{c,r}}\mathcal{A}) \setminus (\operatorname{Con}_{\mathrm{c,r}}\mathcal{B})\}$$

if it exists, ∞ otherwise.

Description of the larder data

Small variations around the following:

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- Here, $\mathrm{dim}(P)$ denotes the order-dimension of P.
- The inequality $\mathrm{crit}_{\mathrm{r}}(\mathcal{A};\mathcal{B}) < \aleph_{\mathrm{dim}(P)-1}$ may hold.

Restricted Kuratowski index of a finite poset

- Actually, $\mathrm{crit}_{\mathrm{r}}(\mathcal{A};\mathcal{B}) \leq \aleph_{\mathrm{kur}_0(P)-1}$, where $\mathrm{kur}_0(P)$, the "restricted Kuratowski index of P", is the least positive integer n such that a certain "existence of large independent sets"-type statement, denoted by $(\aleph_{n-1}, {<}\aleph_0) \rightsquigarrow P$, holds.

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- In particular, calculations of critical points may lead to estimates of the form $\mathrm{crit_r}(\mathcal{A};\mathcal{B}) \leq \aleph_{\log\log n}\dots$

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Theorem (W., 2008)

There exists a non-coordinatizable sectionally complemented modular lattice, of cardinality $\aleph_1$, with a large 4-frame.

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Lattices without congruence-permutable, congruence-preserving extension

An extension $\mathbf{A} \leq \mathbf{B}$ of algebras is congruence-preserving, if the canonical map $\operatorname{Con} \mathbf{A} \to \operatorname{Con} \mathbf{B}$ is an isomorphism.

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Unlike all previous examples, the larder data are difficult to figure out. Let's give an outline.

Semilattice-metric spaces

- A semilattice-metric space is a triple $\mathbf{A} = (A, \delta_{\mathbf{A}}, \tilde{A})$, where A is a set, $\tilde{A}$ is a $(\vee, 0)$-semilattice, $\delta_{\mathbf{A}} \colon A \times A \to \tilde{A}$, $\delta_{\mathbf{A}}(x, x) = 0$, $\delta_{\mathbf{A}}(x, y) = \delta_{\mathbf{A}}(y, x)$, $\delta_{\mathbf{A}}(x, z) \leq \delta_{\mathbf{A}}(x, y) \vee \delta_{\mathbf{A}}(y, z)$ $\forall x, y, z \in A$ (say that $\delta_{\mathbf{A}}$ is a distance).

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- Morphisms: $(f, \tilde{f}) \colon \mathbf{A} \to \mathbf{B}$ means that $f \colon A \to B$, $\tilde{f} \colon \tilde{A} \overset{\vee,0}{\to} \tilde{B}$, and $\delta_{\mathbf{B}}(f(x), f(y)) = \tilde{f}\delta_{\mathbf{A}}(x, y)$ $\forall x, y \in A$.

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- Double arrows in **Metr**: $(f, \tilde{f}) \colon \mathbf{A} \to \mathbf{B}$ such that f is surjective (nothing said about $\tilde{f}$).

Semilattice-metric spaces and covers

- A semilattice-metric cover is a quadruple $\mathbf{A} = (A^*, A, \delta_{\mathbf{A}}, \tilde{A})$, where $A^* \subseteq A$, $(A, \delta_{\mathbf{A}}, \tilde{A})$ is a semilattice-metric space, and $\forall x, y, z \in A^*$, $\exists t \in A$ such that $\delta_{\mathbf{A}}(x, t) \leq \delta_{\mathbf{A}}(y, z)$ and $\delta_{\mathbf{A}}(t, z) \leq \delta_{\mathbf{A}}(x, y)$ (Parallelogram Rule: imitates one step of "congruence-permutable").

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- Morphisms defined as in **Metr**, with $f(A^*) \subseteq B^*$. Get a category $\mathbf{Metr}^*$.

Semilattice-metric spaces and covers

- A semilattice-metric cover is a quadruple $\mathbf{A} = (A^*, A, \delta_{\mathbf{A}}, \tilde{A})$, where $A^* \subseteq A$, $(A, \delta_{\mathbf{A}}, \tilde{A})$ is a semilattice-metric space, and $\forall x, y, z \in A^*$, $\exists t \in A$ such that $\delta_{\mathbf{A}}(x, t) \leq \delta_{\mathbf{A}}(y, z)$ and $\delta_{\mathbf{A}}(t, z) \leq \delta_{\mathbf{A}}(x, y)$ (Parallelogram Rule: imitates one step of "congruence-permutable").
- Morphisms defined as in **Metr**, with $f(A^*) \subseteq B^*$. Get a category $\mathbf{Metr}^*$.
- "Forgetful" functor $\Psi \colon \mathbf{Metr}^* \to \mathbf{Metr}$, $\mathbf{A} \mapsto (A^*, \delta_{\mathbf{A}} \restriction_{A^* \times A^*}, \tilde{A})$.

From algebras to semilattice-metric spaces

- Every algebra $\mathbf{A}$ defines canonically a semilattice-metric space $\Phi(\mathbf{A}) := (A, \operatorname{con}_{\mathbf{A}}, \operatorname{Con}_{\mathrm{c}} \mathbf{A})$, where $\operatorname{con}_{\mathbf{A}}(x, y)$ denotes the (principal) congruence generated by (x, y).

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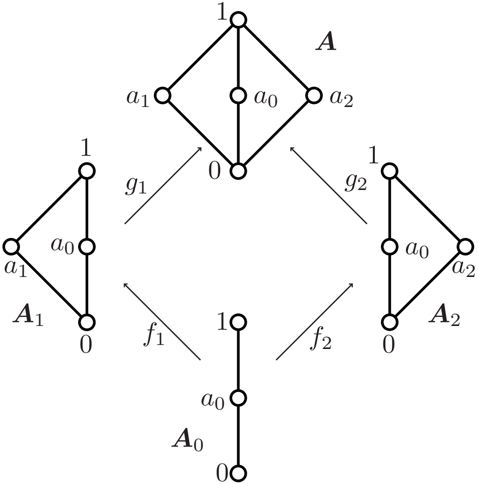
- Then Φ extends naturally to a functor $\mathbf{MAlg} \to \mathbf{Metr}$.

Picture of the larder data

$$\begin{tikzcd}
 & \begin{matrix} \mathcal{S}^{\Rightarrow} := \mathbf{Metr}^{\Rightarrow} \\ \mathcal{S} := \mathbf{Metr} \end{matrix} & \\
\begin{matrix} \mathcal{A} := \mathbf{MAlg} \\ \mathcal{A}^{\dagger} := \mathbf{MAlg}_{\mathrm{fin}} \end{matrix} \arrow[ur, "\Phi"] & & \begin{matrix} \mathcal{B} := \mathbf{Metr}^{*} \\ \mathcal{B}^{\dagger} := \mathbf{Metr}^{*}_{\mathrm{fin}} \end{matrix} \arrow[ul, "\Psi"']
\end{tikzcd}$$

Hard core of the proof 1: a square of finite lattices

The lattices in the two following diagrams have no CPCP-extension that would be functorial wrt. those diagrams:



Hard core of the proof 2: another square of finite lattices