La théorie équationnelle de l'ordre faible de Bruhat

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Outline

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$$\begin{aligned} [N] &\underset{\text{def.}}{=} \{1, 2, \dots, N\}\,, \\ \mathcal{I}_N &\underset{\text{def.}}{=} \{(i, j) \in [N] \times [N] \mid i < j\}\,, \\ \mathsf{inv}(\alpha) &\underset{\text{def.}}{=} \{(i, j) \in \mathcal{I}_N \mid \alpha^{-1}(i) > \alpha^{-1}(j)\}\,. \end{aligned}$$

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- Alternative definition of the permutohedron:

$$\mathsf{P}(N) := \{\mathsf{inv}(\sigma) \mid \sigma \in \mathfrak{S}_N\}\,,\ \text{ordered by } \subseteq\,.$$

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- Conversely, every subset $\boldsymbol{x} \subseteq \mathcal{I}_N$, such that both $\boldsymbol{x}$ and $\mathcal{I}_N \setminus \boldsymbol{x}$ are transitive, is $\mathrm{inv}(\sigma)$ for a unique $\sigma \in \mathfrak{S}_N$ (Dushnik and Miller 1941, Guilbaud and Rosenstiehl 1963).

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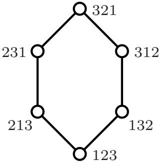
- Hence $\mathrm{P}(N) = \{\boldsymbol{x} \subseteq \mathcal{I}_N \mid \boldsymbol{x} \text{ is clopen}\}$, ordered by $\subseteq$.
- Observe that each $\boldsymbol{x} \in \mathrm{P}(N)$ is a strict ordering. It can be proved (Dushnik and Miller 1941) that those are exactly the finite strict orderings of order-dimension 2.

The permutohedra P(2), P(3), and P(4).

P(2)



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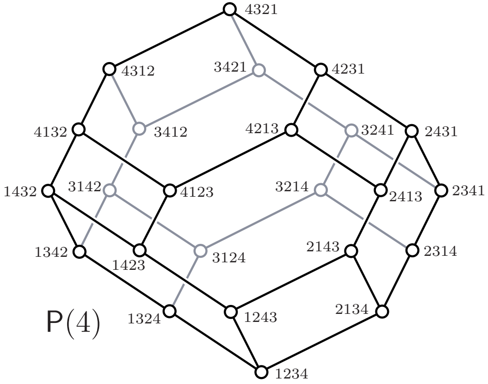

P(4)

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$$\boldsymbol{x} \leq \boldsymbol{y} \Rightarrow \boldsymbol{y}^{\mathsf{c}} \leq \boldsymbol{x}^{\mathsf{c}}\,;$$
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Hence $\mathrm{P}(N)$ is an ortholattice.

What makes P(N) a lattice?

- Every $\boldsymbol{x} \subseteq \mathfrak{I}_N$ is contained in a least closed set, namely, $\mathrm{cl}(\boldsymbol{x}) =$ transitive closure of $\boldsymbol{x}$:

$$\mathrm{cl}(\boldsymbol{x}) = \{(k_0, k_n) \mid k_0 < k_1 < \cdots < k_n\,,\ \text{all } (k_s, k_{s+1}) \in \boldsymbol{x}\}\,.$$

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- Dually, every $\boldsymbol{x} \subseteq \mathfrak{I}_N$ contains a largest open set, namely, $\mathrm{int}(\boldsymbol{x}) = \mathfrak{I}_N \setminus \mathrm{cl}(\mathfrak{I}_N \setminus \boldsymbol{x})$:

$$\mathrm{int}(\boldsymbol{x}) = \{(i, j) \mid \forall i = k_0 < \cdots < k_n = j\,,\ \text{some } (k_s, k_{s+1}) \in \boldsymbol{x}\}\,.$$

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$\mathrm{int}(\boldsymbol{x})$ is closed, for any closed $\boldsymbol{x} \subseteq \mathcal{I}_N$.

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- Likewise, $\boldsymbol{x} \cup \boldsymbol{y}$ is open, and $\boldsymbol{x} \vee \boldsymbol{y} = \mathrm{cl}(\boldsymbol{x} \cup \boldsymbol{y})$.

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The permutohedron $\mathrm{P}(N)$ is McKenzie-bounded, for every positive integer N.

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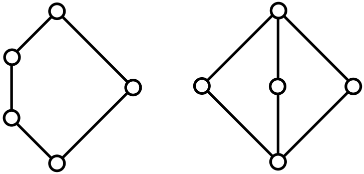

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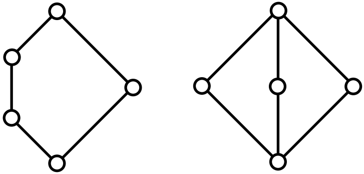

- Every McKenzie-bounded lattice is semidistributive. The converse fails, even for finite lattices.

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Join-irreducibles in $\mathrm{A}_U(N)$ (and $\mathrm{P}(N)$)

- For $(i,j) \in \mathcal{I}_N$, set

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- These are exactly the join-irreducible elements of $\mathrm{A}_U(N)$.
- $(\langle i,j\rangle_U)_* = \langle i,j\rangle_U \setminus \{(i,j)\}$.
- The open subsets of $\mathcal{I}_N$ are exactly the unions of $\langle i,j\rangle_U$.
- The join-irreducible elements of $\mathrm{P}(N)$ are the $\langle i,j\rangle_U$, for $(i,j) \in \mathcal{I}_N$ and $U \subseteq [N]$.

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$\boldsymbol{a}^* = \{(j^*, i^*) \mid (i, j) \in \boldsymbol{a}\}$ (for $\boldsymbol{a} \subseteq \mathcal{I}_N$).

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Set $i^* = N + 1 - i$ (for $i \in [N]$), $U^* = \{i^* \mid i \in U\}$ (for $U \subseteq [N]$), $\boldsymbol{a}^* = \{(j^*, i^*) \mid (i, j) \in \boldsymbol{a}\}$ (for $\boldsymbol{a} \subseteq \mathcal{I}_N$). Then $\boldsymbol{a} \mapsto \boldsymbol{a}^*$ defines an isomorphism from $\mathrm{A}_U(N)$ onto $\mathrm{A}_{[N]\setminus U^*}(N)$.

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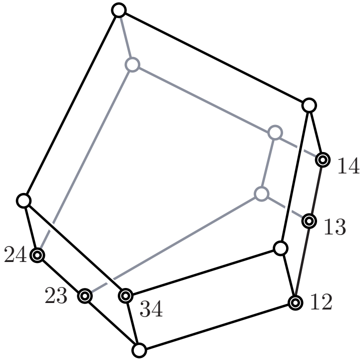
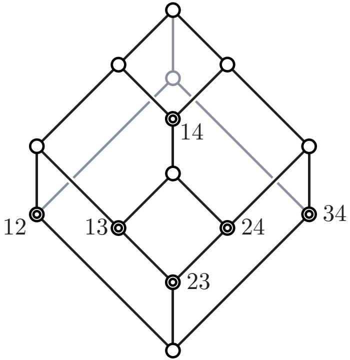
$\psi_U(\boldsymbol{y}) = \{(i,j) \in \mathcal{I}_N \mid \langle i,j \rangle_U \cap \boldsymbol{y} = \varnothing\}$, for all $\boldsymbol{y} \in \mathrm{A}_{U^c}(N)$.

Picturing the Cambrian lattices of type A, for $N = 4$





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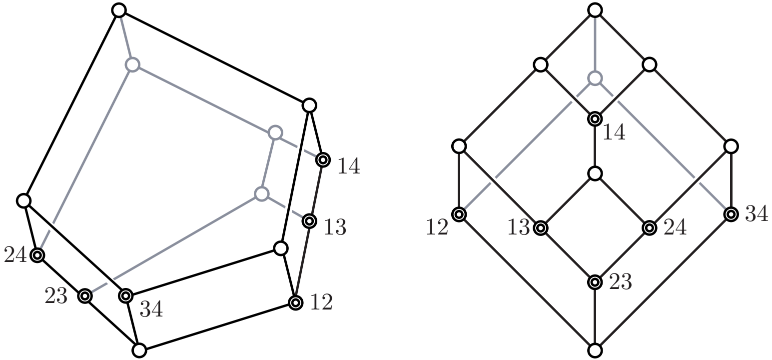

N. Reading observed that each $A_U(N)$ has cardinality $\frac{1}{N+1}\binom{2N}{N}$.

Grätzer's problem for Tamari lattices

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Characterize the (finite) lattices that can be embedded into some Tamari lattice A(N).

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Grätzer's problem is still open: it is still unknown whether

$$\{L \mid (\exists N)(L \hookrightarrow \mathrm{A}(N))\}$$

is decidable.

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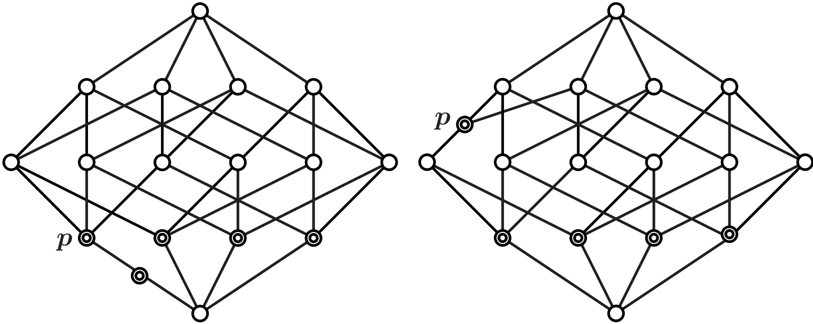
- Conjecture easy to verify for finite distributive lattices.

The lattices B(m, n)



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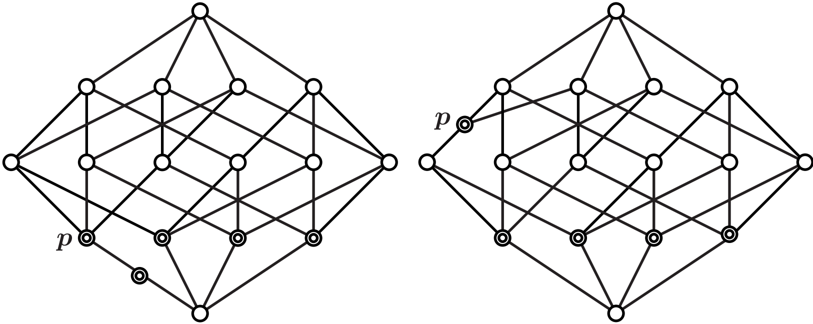

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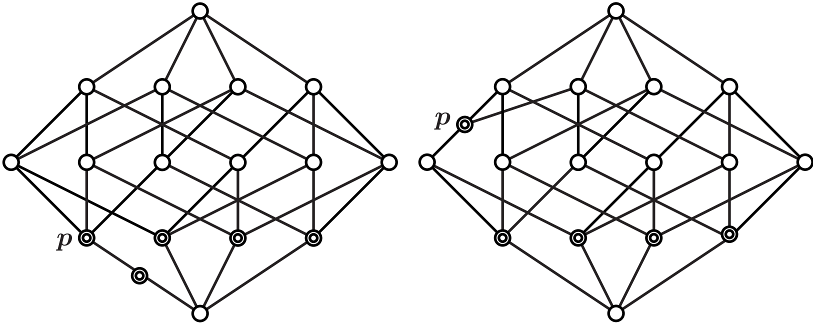

B(1, 3) and B(2, 2), non-atom join-irreducible element is $\boldsymbol{p}$.

- The lattice B(m, n) is defined by doubling the join of m atoms in an $(m+n)$-atom Boolean lattice.
- All lattices B(m, n) are McKenzie-bounded.

B(m, n), A(N), and P(N)

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In particular:

Neither B(2, 2) nor P(4) can be embedded into any A(N) (although they are both McKenzie-bounded).

Vegetables and Gazpachos

- An identity witnessing $\mathrm{B}(2,2) \not\hookrightarrow \mathrm{A}(N)$ is (Veg_1):

$$(\mathrm{a}_1 \vee \mathrm{a}_2 \vee \mathrm{b}_1) \wedge (\mathrm{a}_1 \vee \mathrm{a}_2 \vee \mathrm{b}_2) \leq \bigvee_{i,j \in \{1,2\}} \left((\mathrm{a}_i \vee \tilde{\mathrm{b}}_j) \wedge (\mathrm{a}_1 \vee \mathrm{a}_2 \vee \mathrm{b}_{3-j})\right),$$

with $\tilde{\mathrm{b}}_j = (\mathrm{b}_1 \vee \mathrm{b}_2) \wedge (\mathrm{a}_1 \vee \mathrm{a}_2 \vee \mathrm{b}_j)$,

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- For a finite lattice L, certain U-polarized measures with values in L correspond to lattice embeddings of L into $\mathrm{A}_U(N)$.

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Theorem (S. and W. 2011)

$B(3,3)$ is a homomorphic image of a sublattice of $P(12)$.

- We prove that for a suitable U, the lattice $A_U(12)$ does not satisfy the "splitting identity" of $B(3,3)$:

$$\bigwedge_{1 \leq j \leq 3} (x_1 \vee x_2 \vee x_3 \vee y_j) \leq \bigvee_{1 \leq i \leq 3} (\hat{x}_i \wedge \hat{y}_1 \wedge \hat{y}_2 \wedge \hat{y}_3),$$

where $x = x_1 \vee x_2 \vee x_3$, $y = y_1 \vee y_2 \vee y_3$, $\hat{x}_1 = x_2 \vee x_3 \vee y$, $\hat{y}_1 = y_2 \vee y_3 \vee x$, *etc.*

No separating identity for B(3, 3) (cont'd)

- Relevant values of the x_i, y_i obtained with help of the Prover9-Mace4 program (yields $U = \{5, 6, 9, 10, 11\}$).

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- Variety membership problem, in the $\mathrm{A}_U(N)$, captured by combinatorial objects called scores.
- An (m, n)-score, with respect to $U \subseteq [N]$, expresses a certain tiling property of $m + n$ copies of $[N]$.

Theorem (S. and W. 2014)

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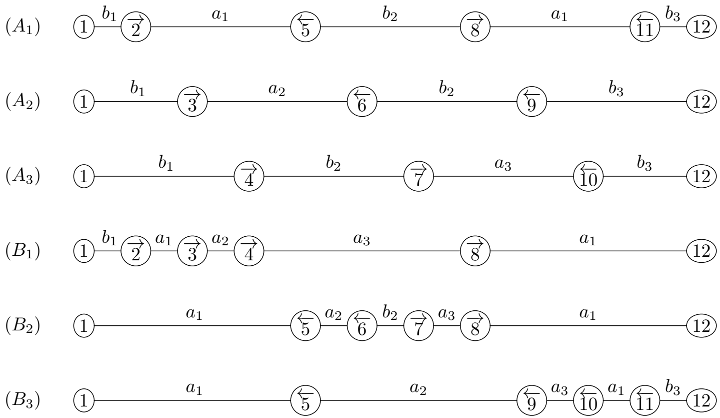
The following statements are equivalent, for all positive integers m, n, N and all $U \subseteq [N]$:

1. $\mathrm{B}(m,n)$ belongs to the lattice variety generated by $\mathrm{A}_U(N)$.
2. $\mathrm{A}_U(N)$ does not satisfy the splitting identity of $\mathrm{B}(m,n)$.
3. There exists an (m,n)-score on $[N]$ with respect to U.

The score for $B(3,3) \in HS(A_U(12))$



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Outline

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- Checking whether $L \in \mathrm{HSP}(\mathcal{X})$ can be difficult.
- An obvious sufficient condition: say that $(\exists X \in \mathcal{X})(\exists e)(e \colon L \hookrightarrow X)$.
- The condition above is not necessary: for example, take $L := \mathrm{B}(3,3)$, $\mathcal{X} := \{\mathrm{P}(n) \mid n \in \mathbb{N}\}$.

Splitting lattices and splitting identities

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The Soprano: Aloysia Weber (1760 – 1839)

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"Born in Zell im Wiesental (Baden-Württemberg, Germany), Aloysia Weber (later on Aloysia Weber-Lange) was one of the four daughters of the musical Weber family."

The Bass: Édouard de Reszke (1853 – 1917)

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"A Polish bass from Warsaw. Born with an impressive natural voice and equipped with compelling histrionic skills, he became one of the most illustrious opera singers active in Europe and America during the late-Victorian era."

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For lattices K and L, a pair (f, g) of maps $K \to L$ is an EA-duet if f is a join-homomorphism, g is a meet-homomorphism, and $f(x) \leq g(y) \Leftrightarrow x \leq y\ \forall x, y \in K$. It is tight if $f = g^{\vee}$ (the largest join-homomorphism $\leq g$) and $g = f^{\wedge}$ (the least meet-homomorphism $\geq f$).

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Strategy for the P(n)

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- We thus need to find a splitting lattice L such that for every (n, U), there is no tight EA-duet $f, g \colon L \to \mathrm{A}_U(n)$.
- Getting at L, and proving that it worked, was the biggest challenge.

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- The bi-ideals form an algebraic lattice.
- $A \otimes B = (\vee, 0)$-semilattice of all compact bi-ideals of $A \times B$.

Useful bi-ideals, universal property

Useful bi-ideals :

- Pure tensors:

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  Belongs to $A \otimes B$ if A and B both have a unit.

- Mixed tensors: $(a \otimes b') \cup (a' \otimes b)$, where $a \leq a'$ and $b \leq b'$.

The box product

Definition (Grätzer and W. 1999)

The box product of lattices A and B, denoted by $A \,\square\, B$, is the set of all finite intersections $\bigcap_{i<n}(a_i \,\square\, b_i)$, where all $(a_i, b_i) \in A \times B$.

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Lemma

Let A and B be finite lattices. If A and B are both McKenzie-bounded (resp., splitting), then so is $A \square B$.

The variety of permutohedra is non-trivial

Theorem (S. and W. 2014)

Let $L := N_5 \,\Box\, B(3,2)$. Then $P(N) \models \theta_L$, for each $N \geq 1$.

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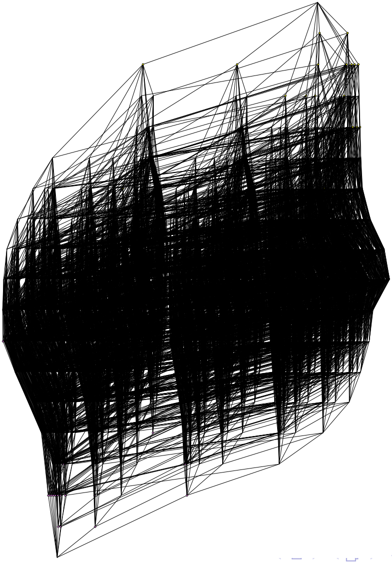
- Brute force computation shows that it has 3,338 elements.
- One needs to prove that there are no (n, U) and no tight EA-duet $f, g \colon \mathrm{N}_5 \,\square\, \mathrm{B}(3,2) \to \mathrm{A}_U(n)$.
- "EA-duet" implies that $f(p \otimes q) \not\subseteq g(p_* \,\square\, q_*)$ (where p and q are the unique join-irreducible, non join-prime elements in N_5 and $\mathrm{B}(3,2)$, respectively); "tight" implies that f and g agree on all join-prime elements of $\mathrm{N}_5 \,\square\, \mathrm{B}(3,2)$.

A portrait view of $N_5 \square B(3,2)$



Outline

The equational theory of permutohedra

The word problem for permutohedra

Given lattice terms s and t, does the relation

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Theorem (S. and W. 2014)

The word problem for permutohedra is decidable.

Pemutohedra and Cambrian lattices

Proposition

For all pair of lattice terms s, t, we have

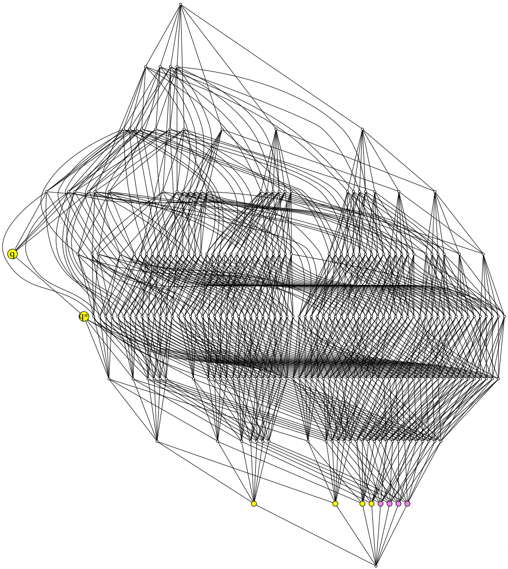
$$P(N) \models s = t \text{ for all } N$$

$$\text{iff}$$

$$A_U(N) \models s = t \text{ for all } N \text{ and } U \subseteq [1, \ldots, N]\,.$$

This is because the Cambrian lattices of type A are the quotients of permutohedra by their minimal meet-irreducible congruences.

The lattice B(4, 4)



The lattices $\mathrm{B}(m, n)$

Recall that the lattice $\mathrm{B}(m, n)$ is obtained from a Boolean algebra over $m + n$ atoms by doubling the join of m atoms.

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Problem

Given m and n, does the lattice $\mathrm{B}(m, n)$ belong to $\mathrm{HSP}(\mathrm{P}(N) \mid N \geq 1)$?

EA-duets and scores

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TFAE:

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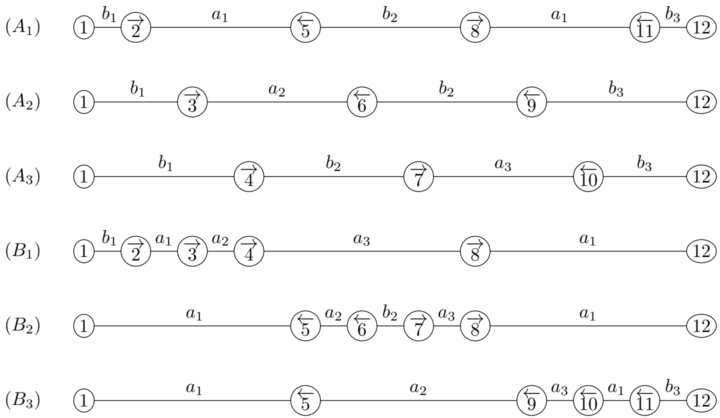
(m,n,N,U)-scores are defined from EA-duets of maps $f, g \colon \mathrm{B}(m,n) \longrightarrow \mathrm{A}_U(N)$, using the isomorphism $\psi_U \colon \mathrm{A}_{U^c}(N) \to \mathrm{A}_U(N)^{\mathrm{op}}$. They express a tiling property of the chain $[N]$.

What does an (m, n, N, U)-score look like?

A $(3, 3, 12, \{5, 6, 9, 10, 11\})$-score :



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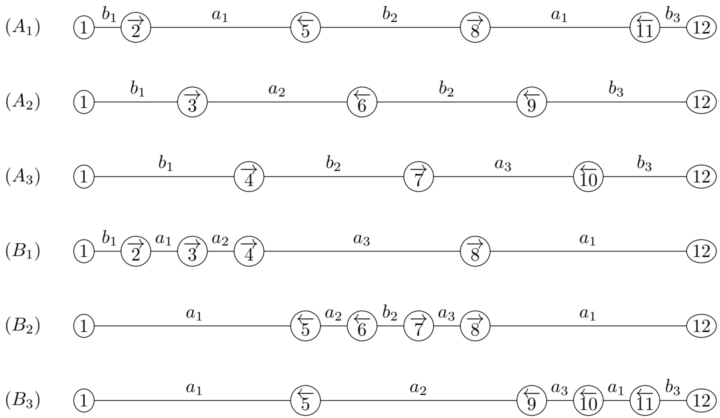

(therefore $B(3, 3) \in HS(P(12))$).

Summarizing

- We can represent a (m, n, N, U)-score via subsets

$$B_i \,,\, A_j \,,\, B_{i,c} \,,\, A_{j,c} \,,$$
$$\text{where } i = 1, \ldots m,\ j = 1, \ldots n\,,\ c \in \{a_1, \ldots, a_n, b_1, \ldots, b_m\},$$

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- We can suppose that B_i , A_j , $B_{i,c}$, $A_{j,c}$ are all subsets of integers (that is, unary [aka monadic] predicates);

- The property

  "$B_i, A_j, B_{i,c}, A_{j,c}$ is an (m, n, N, U)-score"

  is definable in MSO (monadic second order logic of one successor).

MSO, S1S, and Büchi's Theorem

- MSO : atop the first-order language (s) (a unary function symbol), add second-order variables X, Y, Z, ..., and new atomic formulas $t \in X$, where t is a term of (s) and X is a second-order variable.

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Corollary

The problem $\mathrm{B}(m, n) \in \mathrm{HSP}(\mathrm{P}(N) \mid N \geq 1)$ is decidable.

Scores for a pair of terms

Given terms s, t, we can define (within MSO) the concept of an (s, t, N, U)-score, in such a way that:

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Decidability results (S. and W. 2014)

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We can decide whether an identity $s = t$ is satisfied by all Tamari lattices.