Purity and freshness (in categorical model theory)

Friedrich Wehrung

Université de Caen
LMNO, CNRS UMR 6139
Département de Mathématiques
14032 Caen cedex
E-mail: friedrich.wehrung01@unicaen.fr
URL: http://wehrungf.users.lmno.cnrs.fr

September 2020

References

1. J. Adámek and J. Rosický, *Locally Presentable and Accessible Categories*, London Mathematical Society Lecture Notes Series **189**, Cambridge University Press, Cambridge, 1994.

References

1. J. Adámek and J. Rosický, *Locally Presentable and Accessible Categories*, London Mathematical Society Lecture Notes Series **189**, Cambridge University Press, Cambridge, 1994.
2. P. Gillibert and F. Wehrung, *From Objects to Diagrams for Ranges of Functors*, Springer Lecture Notes **2029**, Springer, Heidelberg, 2011.

References

1. J. Adámek and J. Rosický, *Locally Presentable and Accessible Categories*, London Mathematical Society Lecture Notes Series **189**, Cambridge University Press, Cambridge, 1994.
2. P. Gillibert and F. Wehrung, *From Objects to Diagrams for Ranges of Functors*, Springer Lecture Notes **2029**, Springer, Heidelberg, 2011.
3. F. Wehrung, *From non-commutative diagrams to anti-elementary classes*, preprint hal-02000602, J. Math. Logic, to appear.

References

1 J. Adámek and J. Rosický, *Locally Presentable and Accessible Categories*, London Mathematical Society Lecture Notes Series **189**, Cambridge University Press, Cambridge, 1994.

2 P. Gillibert and F. Wehrung, *From Objects to Diagrams for Ranges of Functors*, Springer Lecture Notes **2029**, Springer, Heidelberg, 2011.

3 F. Wehrung, *From non-commutative diagrams to anti-elementary classes*, preprint hal-02000602, J. Math. Logic, to appear.

4 References [2] and [3] above can both be downloaded from https://wehrungf.users.lmno.cnrs.fr/pubs.html .

Motivation

- We would like to prove that certain "naturally defined" categories $\mathcal{C}$ of models (say of first-order theories) are "intractable".

Motivation

- We would like to prove that certain "naturally defined" categories $\mathcal{C}$ of models (say of first-order theories) are "intractable".
- Examples:

Motivation

- We would like to prove that certain "naturally defined" categories $\mathcal{C}$ of models (say of first-order theories) are "intractable".
- Examples: Posets of finitely generated ideals of rings,

Motivation

- We would like to prove that certain "naturally defined" categories $\mathcal{C}$ of models (say of first-order theories) are "intractable".
- Examples: Posets of finitely generated ideals of rings, Ordered K_0 groups of unit-regular rings,

Motivation

- We would like to prove that certain "naturally defined" categories $\mathcal{C}$ of models (say of first-order theories) are "intractable".
- Examples: Posets of finitely generated ideals of rings, Ordered K_0 groups of unit-regular rings, Stone duals of spectra of abelian lattice-ordered groups,

Motivation

- We would like to prove that certain "naturally defined" categories $\mathcal{C}$ of models (say of first-order theories) are "intractable".
- Examples: Posets of finitely generated ideals of rings, Ordered K_0 groups of unit-regular rings, Stone duals of spectra of abelian lattice-ordered groups, …and many other classes.

Motivation

- We would like to prove that certain "naturally defined" categories $\mathcal{C}$ of models (say of first-order theories) are "intractable".
- Examples: Posets of finitely generated ideals of rings, Ordered K_0 groups of unit-regular rings, Stone duals of spectra of abelian lattice-ordered groups, . . . and many other classes.
- A way to define intractability is to state that $\mathcal{C}$ is **not** the class of models of any infinitary (not just first-order!) sentence.

Motivation

- We would like to prove that certain "naturally defined" categories $\mathcal{C}$ of models (say of first-order theories) are "intractable".
- Examples: Posets of finitely generated ideals of rings, Ordered K_0 groups of unit-regular rings, Stone duals of spectra of abelian lattice-ordered groups, ... and many other classes.
- A way to define intractability is to state that $\mathcal{C}$ is **not** the class of models of any infinitary (not just first-order!) sentence.
- In what follows, we introduce a way to construct, for any infinite cardinal λ, an $\mathscr{L}_{\infty\lambda}$-elementary submodel A of a model B, such that $A \in \mathcal{C}$ and $B \notin \mathcal{C}$.

Motivation

- We would like to prove that certain "naturally defined" categories $\mathcal{C}$ of models (say of first-order theories) are "intractable".
- Examples: Posets of finitely generated ideals of rings, Ordered K_0 groups of unit-regular rings, Stone duals of spectra of abelian lattice-ordered groups, ... and many other classes.
- A way to define intractability is to state that $\mathcal{C}$ is **not** the class of models of any infinitary (not just first-order!) sentence.
- In what follows, we introduce a way to construct, for any infinite cardinal λ, an $\mathscr{L}_{\infty\lambda}$-elementary submodel A of a model B, such that $A \in \mathcal{C}$ and $B \notin \mathcal{C}$.
- This proves intractability of $\mathcal{C}$, in the sense above.

Basics

Definition (Prüfer 1923)

A subgroup A of an abelian group B is a pure subgroup if $A \cap nB = nA$ for every integer n.

Basics

Definition (Prüfer 1923)

A subgroup A of an abelian group B is a pure subgroup if $A \cap nB = nA$ for every integer n.

Basic facts:

- Purity means that $\forall a \in A$, $\forall n \in \mathbb{Z} \setminus \{0\}$, if the equation $nx = a$ has a solution in B, then it has one in A.

Basics

Definition (Prüfer 1923)

A subgroup A of an abelian group B is a pure subgroup if $A \cap nB = nA$ for every integer n.

Basic facts:

- Purity means that $\forall a \in A$, $\forall n \in \mathbb{Z} \setminus \{0\}$, if the equation $nx = a$ has a solution in B, then it has one in A.
- If B/A is torsion-free, then A is pure in B

Basics

Definition (Prüfer 1923)

A subgroup A of an abelian group B is a pure subgroup if $A \cap nB = nA$ for every integer n.

Basic facts:

- Purity means that $\forall a \in A$, $\forall n \in \mathbb{Z} \setminus \{0\}$, if the equation $nx = a$ has a solution in B, then it has one in A.
- If B/A is torsion-free, then A is pure in B (*Proof*: let $nb = a$ with $b \in B$; this means that $n(b + A) = 0$ within B/A; since B/A is torsion-free, $b + A = 0$, that is, $b \in A$).

Basics

Definition (Prüfer 1923)

A subgroup A of an abelian group B is a pure subgroup if $A \cap nB = nA$ for every integer n.

Basic facts:

- Purity means that $\forall a \in A$, $\forall n \in \mathbb{Z} \setminus \{0\}$, if the equation $nx = a$ has a solution in B, then it has one in A.
- If B/A is torsion-free, then A is pure in B (*Proof*: let $nb = a$ with $b \in B$; this means that $n(b + A) = 0$ within B/A; since B/A is torsion-free, $b + A = 0$, that is, $b \in A$).
- Every direct summand (equivalently, retract) of an abelian group is a pure subgroup.

Basics

Definition (Prüfer 1923)

A subgroup A of an abelian group B is a pure subgroup if $A \cap nB = nA$ for every integer n.

Basic facts:

- Purity means that $\forall a \in A$, $\forall n \in \mathbb{Z} \setminus \{0\}$, if the equation $nx = a$ has a solution in B, then it has one in A.
- If B/A is torsion-free, then A is pure in B (*Proof*: let $nb = a$ with $b \in B$; this means that $n(b + A) = 0$ within B/A; since B/A is torsion-free, $b + A = 0$, that is, $b \in A$).
- Every direct summand (equivalently, retract) of an abelian group is a pure subgroup. The converse fails.

Finite equation systems and pure subgroups

Finite equation system over an abelian group A:

Finite equation systems and pure subgroups

Finite equation system over an abelian group A:

$$\begin{aligned} k_{11}x_1 + \cdots + k_{1n}x_n &= a_1; \\ k_{21}x_1 + \cdots + k_{2n}x_n &= a_2; \\ \ldots \qquad \ldots \qquad & \ldots \\ k_{m1}x_1 + \cdots + k_{mn}x_n &= a_m, \end{aligned}$$

Finite equation systems and pure subgroups

Finite equation system over an abelian group A:

$$\begin{aligned} k_{11}x_1 + \cdots + k_{1n}x_n &= a_1; \\ k_{21}x_1 + \cdots + k_{2n}x_n &= a_2; \\ \cdots \quad \cdots \quad &\cdots \\ k_{m1}x_1 + \cdots + k_{mn}x_n &= a_m, \end{aligned}$$

with all $k_{ij} \in \mathbb{Z}$ and all $a_i \in A$.

Finite equation systems and pure subgroups

Finite equation system over an abelian group A:

$$\begin{aligned} k_{11}x_1 + \cdots + k_{1n}x_n &= a_1; \\ k_{21}x_1 + \cdots + k_{2n}x_n &= a_2; \\ \cdots \qquad \cdots \qquad &\cdots \\ k_{m1}x_1 + \cdots + k_{mn}x_n &= a_m, \end{aligned}$$

with all $k_{ij} \in \mathbb{Z}$ and all $a_i \in A$.

Theorem (folklore)

A subgroup A is pure in B iff every finite equation system over A, solvable in B, is also solvable in A.

Finite equation systems and pure subgroups

Finite equation system over an abelian group A:

$$\begin{aligned} k_{11}x_1 + \cdots + k_{1n}x_n &= a_1; \\ k_{21}x_1 + \cdots + k_{2n}x_n &= a_2; \\ \cdots \qquad \cdots \qquad & \cdots \\ k_{m1}x_1 + \cdots + k_{mn}x_n &= a_m, \end{aligned}$$

with all $k_{ij} \in \mathbb{Z}$ and all $a_i \in A$.

Theorem (folklore)

A subgroup A is pure in B iff every finite equation system over A, solvable in B, is also solvable in A.

Proof: $\mathbb{Z}$ is a PID, thus, letting $K = (k_{ij})_{i,j}$, there are invertible matrices P and Q such that PKQ is diagonal; this reduces the problem to a system of the form $d_j x_j = a'_j$ $(1 \le j \le n)$, where all $a'_j \in A$ and $d_i \in \mathbb{Z}$.

Σ-structures

- First-order language: $\Sigma = (\mathcal{F}, \mathcal{R}, \mathrm{ar})$ with $\mathcal{F} \cap \mathcal{R} = \varnothing$ and $\mathrm{ar}\colon \mathcal{F} \cup \mathcal{R} \to \mathbb{N}$ with $0 \notin \mathrm{ar}[\mathcal{R}]$.

Σ-structures

- First-order language: $\Sigma = (\mathcal{F}, \mathcal{R}, \mathrm{ar})$ with $\mathcal{F} \cap \mathcal{R} = \varnothing$ and $\mathrm{ar} \colon \mathcal{F} \cup \mathcal{R} \to \mathbb{N}$ with $0 \notin \mathrm{ar}[\mathcal{R}]$. The elements of $\mathcal{F}$ are the function symbols, the elements of $\mathcal{R}$ are the relation symbols, and $\mathrm{ar}(s)$ is the arity of a symbol s. The elements of $\mathcal{F}$ with arity 0 are the constant symbols.

Σ-structures

- First-order language: $\Sigma = (\mathcal{F}, \mathcal{R}, \mathrm{ar})$ with $\mathcal{F} \cap \mathcal{R} = \varnothing$ and $\mathrm{ar}\colon \mathcal{F} \cup \mathcal{R} \to \mathbb{N}$ with $0 \notin \mathrm{ar}[\mathcal{R}]$. The elements of $\mathcal{F}$ are the function symbols, the elements of $\mathcal{R}$ are the relation symbols, and $\mathrm{ar}(s)$ is the arity of a symbol s. The elements of $\mathcal{F}$ with arity 0 are the constant symbols. Add to this an infinite set ("alphabet"), the variables.

Σ-structures

- First-order language: $\Sigma = (\mathcal{F}, \mathcal{R}, \mathrm{ar})$ with $\mathcal{F} \cap \mathcal{R} = \varnothing$ and $\mathrm{ar}\colon \mathcal{F} \cup \mathcal{R} \to \mathbb{N}$ with $0 \notin \mathrm{ar}[\mathcal{R}]$. The elements of $\mathcal{F}$ are the function symbols, the elements of $\mathcal{R}$ are the relation symbols, and $\mathrm{ar}(s)$ is the arity of a symbol s. The elements of $\mathcal{F}$ with arity 0 are the constant symbols. Add to this an infinite set ("alphabet"), the variables.
- model for Σ (or Σ-structure): $\boldsymbol{A} = (A, s^{\boldsymbol{A}})_{s \in \mathcal{F} \cup \mathcal{R}}$, where
  - $R^{\boldsymbol{A}} \subseteq A^{\mathrm{ar}(R)}$ for each $R \in \mathcal{R}$;
  - $f^{\boldsymbol{A}}\colon A^{\mathrm{ar}(f)} \to A$ for each $f \in \mathcal{F}$ with nonzero arity;
  - $c^{\boldsymbol{A}} \in A$ for each constant symbol c.

Σ-structures

- First-order language: $\Sigma = (\mathcal{F}, \mathcal{R}, \mathrm{ar})$ with $\mathcal{F} \cap \mathcal{R} = \varnothing$ and $\mathrm{ar}\colon \mathcal{F} \cup \mathcal{R} \to \mathbb{N}$ with $0 \notin \mathrm{ar}[\mathcal{R}]$. The elements of $\mathcal{F}$ are the function symbols, the elements of $\mathcal{R}$ are the relation symbols, and $\mathrm{ar}(s)$ is the arity of a symbol s. The elements of $\mathcal{F}$ with arity 0 are the constant symbols. Add to this an infinite set ("alphabet"), the variables.
- model for Σ (or Σ-structure): $\boldsymbol{A} = (A, s^{\boldsymbol{A}})_{s \in \mathcal{F} \cup \mathcal{R}}$, where
  - $R^{\boldsymbol{A}} \subseteq A^{\mathrm{ar}(R)}$ for each $R \in \mathcal{R}$;
  - $f^{\boldsymbol{A}}\colon A^{\mathrm{ar}(f)} \to A$ for each $f \in \mathcal{F}$ with nonzero arity;
  - $c^{\boldsymbol{A}} \in A$ for each constant symbol c.
- Terms: closure of variables under all functions symbols.

Σ-structures

- First-order language: $\Sigma = (\mathcal{F}, \mathcal{R}, \mathrm{ar})$ with $\mathcal{F} \cap \mathcal{R} = \varnothing$ and $\mathrm{ar}\colon \mathcal{F} \cup \mathcal{R} \to \mathbb{N}$ with $0 \notin \mathrm{ar}[\mathcal{R}]$. The elements of $\mathcal{F}$ are the function symbols, the elements of $\mathcal{R}$ are the relation symbols, and $\mathrm{ar}(s)$ is the arity of a symbol s. The elements of $\mathcal{F}$ with arity 0 are the constant symbols. Add to this an infinite set ("alphabet"), the variables.
- model for Σ (or Σ-structure): $\boldsymbol{A} = (A, s^{\boldsymbol{A}})_{s \in \mathcal{F} \cup \mathcal{R}}$, where
  - $R^{\boldsymbol{A}} \subseteq A^{\mathrm{ar}(R)}$ for each $R \in \mathcal{R}$;
  - $f^{\boldsymbol{A}}\colon A^{\mathrm{ar}(f)} \to A$ for each $f \in \mathcal{F}$ with nonzero arity;
  - $c^{\boldsymbol{A}} \in A$ for each constant symbol c.
- Terms: closure of variables under all functions symbols.
- atomic formulas: $Rt_1 \dots t_n$, where the t_i are terms and R is either the equality ($n = 2$), or a relation symbol ($n = \mathrm{ar}(R)$).

Σ-structures

- First-order language: $\Sigma = (\mathcal{F}, \mathcal{R}, \mathrm{ar})$ with $\mathcal{F} \cap \mathcal{R} = \varnothing$ and $\mathrm{ar}\colon \mathcal{F} \cup \mathcal{R} \to \mathbb{N}$ with $0 \notin \mathrm{ar}[\mathcal{R}]$. The elements of $\mathcal{F}$ are the function symbols, the elements of $\mathcal{R}$ are the relation symbols, and $\mathrm{ar}(s)$ is the arity of a symbol s. The elements of $\mathcal{F}$ with arity 0 are the constant symbols. Add to this an infinite set ("alphabet"), the variables.
- model for Σ (or Σ-structure): $\boldsymbol{A} = (A, s^{\boldsymbol{A}})_{s \in \mathcal{F} \cup \mathcal{R}}$, where
  - $R^{\boldsymbol{A}} \subseteq A^{\mathrm{ar}(R)}$ for each $R \in \mathcal{R}$;
  - $f^{\boldsymbol{A}}\colon A^{\mathrm{ar}(f)} \to A$ for each $f \in \mathcal{F}$ with nonzero arity;
  - $c^{\boldsymbol{A}} \in A$ for each constant symbol c.
- Terms: closure of variables under all functions symbols.
- atomic formulas: $Rt_1 \ldots t_n$, where the t_i are terms and R is either the equality ($n = 2$), or a relation symbol ($n = \mathrm{ar}(R)$).
- First-order formulas obtained by closing the atomic formulas under $\wedge$, $\neg$, $\exists$ (thus also $\vee$, $\forall$).

Σ-structures (cont'd)

- Formulas with parameters from a model $\boldsymbol{A}$: some free variables of φ are assigned to elements of A.

Σ-structures (cont'd)

- Formulas with parameters from a model $\boldsymbol{A}$: some free variables of φ are assigned to elements of A.
- Satisfaction of a closed (no free variables) formula φ with parameters from a model $\boldsymbol{A}$ (in symbol $\boldsymbol{A} \models \varphi$) defined by induction on the complexity of φ, the usual way.

Σ-structures (cont'd)

- Formulas with parameters from a model $\boldsymbol{A}$: some free variables of φ are assigned to elements of A.
- Satisfaction of a closed (no free variables) formula φ with parameters from a model $\boldsymbol{A}$ (in symbol $\boldsymbol{A} \models \varphi$) defined by induction on the complexity of φ, the usual way.
- Elementary equivalence of models $\boldsymbol{A}$ and $\boldsymbol{B}$: $\boldsymbol{A} \models \varphi$ iff $\boldsymbol{B} \models \varphi$, for every closed formula φ.

Σ-structures (cont'd)

- Formulas with parameters from a model $\boldsymbol{A}$: some free variables of φ are assigned to elements of A.
- Satisfaction of a closed (no free variables) formula φ with parameters from a model $\boldsymbol{A}$ (in symbol $\boldsymbol{A} \models \varphi$) defined by induction on the complexity of φ, the usual way.
- Elementary equivalence of models $\boldsymbol{A}$ and $\boldsymbol{B}$: $\boldsymbol{A} \models \varphi$ iff $\boldsymbol{B} \models \varphi$, for every closed formula φ.
- A map $f\colon A \to B$ is an elementary embedding if $\boldsymbol{A} \models \varphi(\vec{a}) \Leftrightarrow \boldsymbol{B} \models \varphi(f\vec{a})$, for every closed formula φ with parameters $\vec{a}$ from $\boldsymbol{A}$.

Σ-structures (cont'd)

- Formulas with parameters from a model $\boldsymbol{A}$: some free variables of φ are assigned to elements of A.
- Satisfaction of a closed (no free variables) formula φ with parameters from a model $\boldsymbol{A}$ (in symbol $\boldsymbol{A} \models \varphi$) defined by induction on the complexity of φ, the usual way.
- Elementary equivalence of models $\boldsymbol{A}$ and $\boldsymbol{B}$: $\boldsymbol{A} \models \varphi$ iff $\boldsymbol{B} \models \varphi$, for every closed formula φ.
- A map $f: A \to B$ is an elementary embedding if $\boldsymbol{A} \models \varphi(\vec{a}) \Leftrightarrow \boldsymbol{B} \models \varphi(f\vec{a})$, for every closed formula φ with parameters $\vec{a}$ from $\boldsymbol{A}$.
- Direction $\Rightarrow$ (resp., $\Leftrightarrow$) for atomic φ: we say that f is a homomorphism (resp., an embedding).

Σ-structures (cont'd)

- Formulas with parameters from a model $\boldsymbol{A}$: some free variables of φ are assigned to elements of A.
- Satisfaction of a closed (no free variables) formula φ with parameters from a model $\boldsymbol{A}$ (in symbol $\boldsymbol{A} \models \varphi$) defined by induction on the complexity of φ, the usual way.
- Elementary equivalence of models $\boldsymbol{A}$ and $\boldsymbol{B}$: $\boldsymbol{A} \models \varphi$ iff $\boldsymbol{B} \models \varphi$, for every closed formula φ.
- A map $f \colon A \to B$ is an elementary embedding if $\boldsymbol{A} \models \varphi(\vec{a}) \Leftrightarrow \boldsymbol{B} \models \varphi(f\vec{a})$, for every closed formula φ with parameters $\vec{a}$ from $\boldsymbol{A}$.
- Direction $\Rightarrow$ (resp., $\Leftrightarrow$) for atomic φ: we say that f is a homomorphism (resp., an embedding).
- Those concepts can be extended to many logics (special sets of formulas, infinitary logics...).

Purity for Σ-structures

- An atomic system over a model $\boldsymbol{A}$ is a set of atomic formulas with parameters from $\boldsymbol{A}$.

Purity for Σ-structures

- An atomic system over a model $\boldsymbol{A}$ is a set of atomic formulas with parameters from $\boldsymbol{A}$.
- For any homomorphism $f\colon \boldsymbol{A} \to \boldsymbol{B}$, if an atomic system $\Phi(\vec{a})$ over $\boldsymbol{A}$ has a solution over $\boldsymbol{A}$, then $\Phi(f\vec{a})$ has a solution over $\boldsymbol{B}$ (*straightforward*).

Purity for Σ-structures

- An atomic system over a model $\boldsymbol{A}$ is a set of atomic formulas with parameters from $\boldsymbol{A}$.
- For any homomorphism $f\colon \boldsymbol{A} \to \boldsymbol{B}$, if an atomic system $\Phi(\vec{a})$ over $\boldsymbol{A}$ has a solution over $\boldsymbol{A}$, then $\Phi(f\vec{a})$ has a solution over $\boldsymbol{B}$ (*straightforward*).
- A homomorphism $f\colon \boldsymbol{A} \to \boldsymbol{B}$ is pure if for every finite atomic system $\Phi(\vec{a})$ over $\boldsymbol{A}$, if $\Phi(f\vec{a})$ is solvable in $\boldsymbol{B}$, then $\Phi(\vec{a})$ is solvable in $\boldsymbol{A}$.

Purity for Σ-structures

- An atomic system over a model $\boldsymbol{A}$ is a set of atomic formulas with parameters from $\boldsymbol{A}$.
- For any homomorphism $f\colon \boldsymbol{A} \to \boldsymbol{B}$, if an atomic system $\Phi(\vec{a})$ over $\boldsymbol{A}$ has a solution over $\boldsymbol{A}$, then $\Phi(f\vec{a})$ has a solution over $\boldsymbol{B}$ (*straightforward*).
- A homomorphism $f\colon \boldsymbol{A} \to \boldsymbol{B}$ is pure if for every finite atomic system $\Phi(\vec{a})$ over $\boldsymbol{A}$, if $\Phi(f\vec{a})$ is solvable in $\boldsymbol{B}$, then $\Phi(\vec{a})$ is solvable in $\boldsymbol{A}$.
- Equivalently: for every finite conjunction φ of atomic formulas, $\boldsymbol{B} \models \exists\vec{x}\varphi(f\vec{a},\vec{x})$ implies $\boldsymbol{A} \models \exists\vec{x}\varphi(\vec{a},\vec{x})$.

Purity for Σ-structures

- An atomic system over a model $\boldsymbol{A}$ is a set of atomic formulas with parameters from $\boldsymbol{A}$.
- For any homomorphism $f\colon \boldsymbol{A} \to \boldsymbol{B}$, if an atomic system $\Phi(\vec{a})$ over $\boldsymbol{A}$ has a solution over $\boldsymbol{A}$, then $\Phi(f\vec{a})$ has a solution over $\boldsymbol{B}$ (*straightforward*).
- A homomorphism $f\colon \boldsymbol{A} \to \boldsymbol{B}$ is pure if for every finite atomic system $\Phi(\vec{a})$ over $\boldsymbol{A}$, if $\Phi(f\vec{a})$ is solvable in $\boldsymbol{B}$, then $\Phi(\vec{a})$ is solvable in $\boldsymbol{A}$.
- Equivalently: for every finite conjunction φ of atomic formulas, $\boldsymbol{B} \models \exists\vec{x}\varphi(f\vec{a},\vec{x})$ implies $\boldsymbol{A} \models \exists\vec{x}\varphi(\vec{a},\vec{x})$.
- For abelian groups, we recover the usual concept of purity.

A categorical formulation of purity

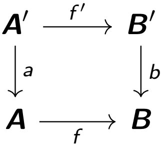
Proposition (Adámek and Rosický 1994)

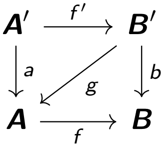
Let Σ be a finite first-order language. An embedding $f\colon \boldsymbol{A} \to \boldsymbol{B}$, of models of Σ, is pure iff for all finitely presentable $\boldsymbol{A}'$ and $\boldsymbol{B}'$ and all homomorphisms $a\colon \boldsymbol{A}' \to \boldsymbol{A}$, $b\colon \boldsymbol{B}' \to \boldsymbol{B}$, and $f'\colon \boldsymbol{A}' \to \boldsymbol{B}'$, if $f \circ a = b \circ f'$, then there exists a homomorphism $g\colon \boldsymbol{B}' \to \boldsymbol{A}$ such that $a = g \circ f'$.

A categorical formulation of purity

Proposition (Adámek and Rosický 1994)

Let Σ be a finite first-order language. An embedding $f\colon \boldsymbol{A} \to \boldsymbol{B}$, of models of Σ, is pure iff for all finitely presentable $\boldsymbol{A}'$ and $\boldsymbol{B}'$ and all homomorphisms $a\colon \boldsymbol{A}' \to \boldsymbol{A}$, $b\colon \boldsymbol{B}' \to \boldsymbol{B}$, and $f'\colon \boldsymbol{A}' \to \boldsymbol{B}'$, if $f \circ a = b \circ f'$, then there exists a homomorphism $g\colon \boldsymbol{B}' \to \boldsymbol{A}$ such that $a = g \circ f'$.





A categorical formulation of purity

Proposition (Adámek and Rosický 1994)

Let Σ be a finite first-order language. An embedding $f\colon \boldsymbol{A} \to \boldsymbol{B}$, of models of Σ, is pure iff for all finitely presentable $\boldsymbol{A}'$ and $\boldsymbol{B}'$ and all homomorphisms $a\colon \boldsymbol{A}' \to \boldsymbol{A}$, $b\colon \boldsymbol{B}' \to \boldsymbol{B}$, and $f'\colon \boldsymbol{A}' \to \boldsymbol{B}'$, if $f \circ a = b \circ f'$, then there exists a homomorphism $g\colon \boldsymbol{B}' \to \boldsymbol{A}$ such that $a = g \circ f'$.

$$\begin{array}{ccc} \boldsymbol{A}' & \xrightarrow{f'} & \boldsymbol{B}' \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ \boldsymbol{A} & \xrightarrow[f]{} & \boldsymbol{B} \end{array} \qquad\qquad \begin{array}{ccc} \boldsymbol{A}' & \xrightarrow{f'} & \boldsymbol{B}' \\ {\scriptstyle a}\downarrow & \swarrow{\scriptstyle g} & \downarrow{\scriptstyle b} \\ \boldsymbol{A} & \xrightarrow[f]{} & \boldsymbol{B} \end{array}$$

A model $\boldsymbol{C}$ is finitely presentable if for every directed colimit $S = \varinjlim_{i\in I} S_i$, every homomorphism $c\colon C \to S$ factors "in an essentially unique way" through some S_i.

λ-purity in an arbitrary category

- Purity is thus a categorical concept.

λ-purity in an arbitrary category

- Purity is thus a categorical concept.
- Can be extended to λ-purity, for any infinite regular cardinal λ (so purity is just ω-purity).

λ-purity in an arbitrary category

- Purity is thus a categorical concept.
- Can be extended to λ-purity, for any infinite regular cardinal λ (so purity is just ω-purity).

Definition (Adámek and Rosický 1994)

Let $\mathcal{C}$ be a category and let λ be an infinite regular cardinal. A morphism $f\colon A \to B$ in $\mathcal{C}$ is λ-pure iff for all λ-presentable A' and B' and all morphisms $a\colon A' \to A$, $b\colon B' \to B$, and $f'\colon A' \to B'$, if $f \circ a = b \circ f'$, then there exists a morphism $g\colon B' \to A$ such that $a = g \circ f'$.

λ-purity in an arbitrary category

- Purity is thus a categorical concept.
- Can be extended to λ-purity, for any infinite regular cardinal λ (so purity is just ω-purity).

Definition (Adámek and Rosický 1994)

Let $\mathcal{C}$ be a category and let λ be an infinite regular cardinal. A morphism $f\colon A \to B$ in $\mathcal{C}$ is λ-pure iff for all λ-presentable A' and B' and all morphisms $a\colon A' \to A$, $b\colon B' \to B$, and $f'\colon A' \to B'$, if $f \circ a = b \circ f'$, then there exists a morphism $g\colon B' \to A$ such that $a = g \circ f'$.

- Can be expressed in terms of λ-small atomic systems.

λ-purity in an arbitrary category

- Purity is thus a categorical concept.
- Can be extended to λ-purity, for any infinite regular cardinal λ (so purity is just ω-purity).

Definition (Adámek and Rosický 1994)

Let $\mathcal{C}$ be a category and let λ be an infinite regular cardinal. A morphism $f\colon A \to B$ in $\mathcal{C}$ is λ-pure iff for all λ-presentable A' and B' and all morphisms $a\colon A' \to A$, $b\colon B' \to B$, and $f'\colon A' \to B'$, if $f \circ a = b \circ f'$, then there exists a morphism $g\colon B' \to A$ such that $a = g \circ f'$.

- Can be expressed in terms of λ-small atomic systems.
- λ-presentability defined the same way as finite presentability, now with I λ-directed (every λ-small subset has an upper bound) instead of just directed

λ-purity in an arbitrary category

- Purity is thus a categorical concept.
- Can be extended to λ-purity, for any infinite regular cardinal λ (so purity is just ω-purity).

Definition (Adámek and Rosický 1994)

Let $\mathcal{C}$ be a category and let λ be an infinite regular cardinal. A morphism $f\colon A \to B$ in $\mathcal{C}$ is λ-pure iff for all λ-presentable A' and B' and all morphisms $a\colon A' \to A$, $b\colon B' \to B$, and $f'\colon A' \to B'$, if $f \circ a = b \circ f'$, then there exists a morphism $g\colon B' \to A$ such that $a = g \circ f'$.

- Can be expressed in terms of λ-small atomic systems.
- λ-presentability defined the same way as finite presentability, now with I λ-directed (every λ-small subset has an upper bound) instead of just directed (so λ-purity gets stronger as λ increases).

λ-freshness

Even "∞-purity" of $f\colon A \to B$ just means that f has a retraction (i.e., $(\exists g)(g \circ f = \mathrm{id})$),

λ-freshness

Even "∞-purity" of $f\colon A \to B$ just means that f has a retraction (i.e., $(\exists g)(g \circ f = \mathrm{id})$), which does not imply that f is an elementary embedding.

λ-freshness

Even "∞-purity" of $f : A \to B$ just means that f has a retraction (i.e., $(\exists g)(g \circ f = \mathrm{id})$), which does not imply that f is an elementary embedding. We need a stronger concept.

λ-freshness

Even "∞-purity" of $f: A \to B$ just means that f has a retraction (i.e., $(\exists g)(g \circ f = \mathrm{id})$), which does not imply that f is an elementary embedding. We need a stronger concept.

Definition (W 2019)

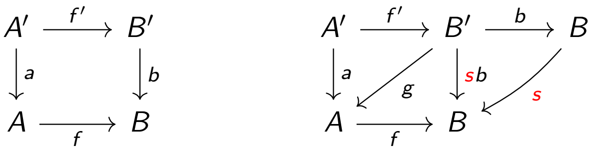
Let $\mathcal{C}$ be a category and let λ be an infinite regular cardinal. A morphism $f: A \to B$ in $\mathcal{C}$ is λ-fresh if for all λ-presentable A' and B' and all morphisms $a: A' \to A$, $b: B' \to B$, and $f': A' \to B'$, if $f \circ a = b \circ f'$, then there are an automorphism s of B and a morphism $g: B' \to A$ such that $a = g \circ f'$ and $f \circ g = s \circ b$.

λ-freshness

Even "∞-purity" of $f\colon A \to B$ just means that f has a retraction (i.e., $(\exists g)(g \circ f = \mathrm{id})$), which does not imply that f is an elementary embedding. We need a stronger concept.

Definition (W 2019)

Let $\mathcal{C}$ be a category and let λ be an infinite regular cardinal. A morphism $f\colon A \to B$ in $\mathcal{C}$ is λ-fresh if for all λ-presentable A' and B' and all morphisms $a\colon A' \to A$, $b\colon B' \to B$, and $f'\colon A' \to B'$, if $f \circ a = b \circ f'$, then there are an automorphism s of B and a morphism $g\colon B' \to A$ such that $a = g \circ f'$ and $f \circ g = s \circ b$.



The symmetric category $\mathfrak{P}_{\mathrm{inj}}(\Omega)$

Definition

Define the symmetric category over a set Ω, and denote it by $\mathfrak{P}_{\mathrm{inj}}(\Omega)$, by the one whose objects are the subsets of Ω, and whose morphisms are the one-to-one maps $f\colon X \rightarrowtail Y$ where $X, Y \subseteq \Omega$.

The symmetric category $\mathfrak{P}_{\mathrm{inj}}(\Omega)$

Definition

Define the symmetric category over a set Ω, and denote it by $\mathfrak{P}_{\mathrm{inj}}(\Omega)$, by the one whose objects are the subsets of Ω, and whose morphisms are the one-to-one maps $f\colon X \rightarrowtail Y$ where $X, Y \subseteq \Omega$.

- The only compositions occurring in $\mathfrak{P}_{\mathrm{inj}}(\Omega)$ are the $g \circ f$, where $f\colon X \rightarrowtail Y$ and $g\colon Y \rightarrowtail Z$ (and $(g \circ f)(x) = g(f(x))$ as usual).

The symmetric category $\mathfrak{P}_{\mathrm{inj}}(\Omega)$

Definition

Define the symmetric category over a set Ω, and denote it by $\mathfrak{P}_{\mathrm{inj}}(\Omega)$, by the one whose objects are the subsets of Ω, and whose morphisms are the one-to-one maps $f\colon X \rightarrowtail Y$ where $X, Y \subseteq \Omega$.

- The only compositions occurring in $\mathfrak{P}_{\mathrm{inj}}(\Omega)$ are the $g \circ f$, where $f\colon X \rightarrowtail Y$ and $g\colon Y \rightarrowtail Z$ (and $(g \circ f)(x) = g(f(x))$ as usual).
- Directed colimits in $\mathfrak{P}_{\mathrm{inj}}(\Omega)$: $X = \varinjlim_{i \in I} X_i$ means that up to isomorphism, $X = \bigcup_{i \in I} X_i$ (directed union).

The symmetric category $\mathfrak{P}_{\mathrm{inj}}(\Omega)$

Definition

Define the symmetric category over a set Ω, and denote it by $\mathfrak{P}_{\mathrm{inj}}(\Omega)$, by the one whose objects are the subsets of Ω, and whose morphisms are the one-to-one maps $f\colon X \rightarrowtail Y$ where $X, Y \subseteq \Omega$.

- The only compositions occurring in $\mathfrak{P}_{\mathrm{inj}}(\Omega)$ are the $g \circ f$, where $f\colon X \rightarrowtail Y$ and $g\colon Y \rightarrowtail Z$ (and $(g \circ f)(x) = g(f(x))$ as usual).
- Directed colimits in $\mathfrak{P}_{\mathrm{inj}}(\Omega)$: $X = \varinjlim_{i \in I} X_i$ means that up to isomorphism, $X = \bigcup_{i \in I} X_i$ (directed union).
- A subset $X \subseteq \Omega$ is λ-presentable, within $\mathfrak{P}_{\mathrm{inj}}(\Omega)$, iff $\operatorname{card} X < \lambda$.

Purity and freshness on symmetric categories

Proposition (W 2019)

Let λ be an infinite regular cardinal, let Ω be a set, and let $f\colon A \rightarrowtail B$ in $\mathfrak{P}_{\mathrm{inj}}(\Omega)$. The following are equivalent:

1. f is λ-fresh.
2. f is λ-pure.
3. Either f is a bijection or $\lambda \leq \operatorname{card} A$.

Purity and freshness on symmetric categories

Proposition (W 2019)

Let λ be an infinite regular cardinal, let Ω be a set, and let $f\colon A \rightarrowtail B$ in $\mathfrak{P}_{\mathrm{inj}}(\Omega)$. The following are equivalent:

1. f is λ-fresh.
2. f is λ-pure.
3. Either f is a bijection or $\lambda \leq \operatorname{card} A$.

Outline of proof of the interesting direction

Purity and freshness on symmetric categories

Proposition (W 2019)

Let λ be an infinite regular cardinal, let Ω be a set, and let $f\colon A \rightarrowtail B$ in $\mathfrak{P}_{\mathrm{inj}}(\Omega)$. The following are equivalent:

1. f is λ-fresh.
2. f is λ-pure.
3. Either f is a bijection or $\lambda \leq \operatorname{card} A$.

Outline of proof of the interesting direction (3)⇒(1):

Purity and freshness on symmetric categories

Proposition (W 2019)

Let λ be an infinite regular cardinal, let Ω be a set, and let $f\colon A \rightarrowtail B$ in $\mathfrak{P}_{\mathrm{inj}}(\Omega)$. The following are equivalent:

1. f is λ-fresh.
2. f is λ-pure.
3. Either f is a bijection or $\lambda \leq \operatorname{card} A$.

Outline of proof of the interesting direction (3)$\Rightarrow$(1):

- Let us suppose that f is the inclusion map $A \hookrightarrow B$ with $\lambda \leq \operatorname{card} A$.

Purity and freshness on symmetric categories

Proposition (W 2019)

Let λ be an infinite regular cardinal, let Ω be a set, and let $f\colon A \rightarrowtail B$ in $\mathfrak{P}_{\mathrm{inj}}(\Omega)$. The following are equivalent:

1. f is λ-fresh.
2. f is λ-pure.
3. Either f is a bijection or $\lambda \leq \operatorname{card} A$.

Outline of proof of the interesting direction (3)$\Rightarrow$(1):

- Let us suppose that f is the inclusion map $A \hookrightarrow B$ with $\lambda \leq \operatorname{card} A$.
- Given λ-small $B' \subset B$ and $A' \subseteq A \cap B'$, we must find a permutation σ of B such that $\sigma\restriction_{A'} = \mathrm{id}$ and $\sigma B' \subseteq A$.

Purity and freshness on symmetric categories

Proposition (W 2019)

Let λ be an infinite regular cardinal, let Ω be a set, and let $f\colon A \rightarrowtail B$ in $\mathfrak{P}_{\mathrm{inj}}(\Omega)$. The following are equivalent:

1. f is λ-fresh.
2. f is λ-pure.
3. Either f is a bijection or $\lambda \leq \operatorname{card} A$.

Outline of proof of the interesting direction (3)⇒(1):

- Let us suppose that f is the inclusion map $A \hookrightarrow B$ with $\lambda \leq \operatorname{card} A$.
- Given λ-small $B' \subset B$ and $A' \subseteq A \cap B'$, we must find a permutation σ of B such that $\sigma\restriction_{A'} = \mathrm{id}$ and $\sigma B' \subseteq A$.
- Since $\operatorname{card} B' < \lambda \leq \operatorname{card} A$, there is enough room in A for this.

Infinitary languages

- For cardinals κ and λ with $\kappa \geq \lambda$, infinitary formulas (denoted $\mathscr{L}_{\kappa\lambda}$) over a first-order language Σ are defined the same way as ordinary first-order formulas, with the following differences:

Infinitary languages

- For cardinals κ and λ with $\kappa \geq \lambda$, infinitary formulas (denoted $\mathscr{L}_{\kappa\lambda}$) over a first-order language Σ are defined the same way as ordinary first-order formulas, with the following differences:
-
  - For any $\alpha < \kappa$ and any collection $\{\varphi_\xi \mid \xi < \alpha\}$ of formulas in $\mathscr{L}_{\kappa\lambda}$, over less than λ free variables altogether, the conjunction $\bigwedge_{\xi<\alpha} \varphi_\xi$ is a formula in $\mathscr{L}_{\kappa\lambda}$.
  - For any family $(x_\eta \mid \eta < \beta)$, with $\beta < \lambda$, of free variables of a formula $\varphi \in \mathscr{L}_{\kappa\lambda}$, the formula $(\exists_{\eta<\beta} x_\eta)\varphi$ is in $\mathscr{L}_{\kappa\lambda}$.

Infinitary languages

- For cardinals κ and λ with $\kappa \geq \lambda$, infinitary formulas (denoted $\mathscr{L}_{\kappa\lambda}$) over a first-order language Σ are defined the same way as ordinary first-order formulas, with the following differences:
- — For any $\alpha < \kappa$ and any collection $\{\varphi_\xi \mid \xi < \alpha\}$ of formulas in $\mathscr{L}_{\kappa\lambda}$, over less than λ free variables altogether, the conjunction $\bigwedge_{\xi<\alpha} \varphi_\xi$ is a formula in $\mathscr{L}_{\kappa\lambda}$.
  — For any family $(x_\eta \mid \eta < \beta)$, with $\beta < \lambda$, of free variables of a formula $\varphi \in \mathscr{L}_{\kappa\lambda}$, the formula $(\exists_{\eta<\beta} x_\eta)\varphi$ is in $\mathscr{L}_{\kappa\lambda}$.
- Extension to $\kappa = \infty$ (yields $\mathscr{L}_{\infty\lambda}$): we allow arbitrary conjunctions on sets of formulas over less than λ free variables altogether, and λ-small quantifications.

Infinitary languages

- For cardinals κ and λ with $\kappa \geq \lambda$, infinitary formulas (denoted $\mathscr{L}_{\kappa\lambda}$) over a first-order language Σ are defined the same way as ordinary first-order formulas, with the following differences:
- — For any $\alpha < \kappa$ and any collection $\{\varphi_\xi \mid \xi < \alpha\}$ of formulas in $\mathscr{L}_{\kappa\lambda}$, over less than λ free variables altogether, the conjunction $\bigwedge_{\xi<\alpha}\varphi_\xi$ is a formula in $\mathscr{L}_{\kappa\lambda}$.
  — For any family $(x_\eta \mid \eta < \beta)$, with $\beta < \lambda$, of free variables of a formula $\varphi \in \mathscr{L}_{\kappa\lambda}$, the formula $(\exists_{\eta<\beta}x_\eta)\varphi$ is in $\mathscr{L}_{\kappa\lambda}$.
- Extension to $\kappa = \infty$ (yields $\mathscr{L}_{\infty\lambda}$): we allow arbitrary conjunctions on sets of formulas over less than λ free variables altogether, and λ-small quantifications.
- Ordinary first-order logic: it is $\mathscr{L}_{\omega\omega}$.

Examples outside first-order logic

- Finiteness (of the ambiant universe) is $\mathscr{L}_{\omega_1\omega}$:

$$\bigvee\!\!\!\bigvee_{n<\omega}(\exists_{i<n}x_i)(\forall x)\bigvee\!\!\!\bigvee_{i<n}(x=x_i)\,.$$

Examples outside first-order logic

- Finiteness (of the ambiant universe) is $\mathscr{L}_{\omega_1\omega}$:

$$\bigvee\!\!\!\bigvee_{n<\omega}(\exists_{i<n}x_i)(\forall x)\bigvee\!\!\!\bigvee_{i<n}(x=x_i)\,.$$

- Well-foundedness (of the ambiant poset) is $\mathscr{L}_{\omega_1\omega_1}$:

$$(\forall_{n<\omega}x_n)\bigvee\!\!\!\bigvee_{n<\omega}(x_{n+1}\not<x_n)\,.$$

Examples outside first-order logic

- Finiteness (of the ambiant universe) is $\mathscr{L}_{\omega_1\omega}$:

$$\bigvee\!\!\!\bigvee_{n<\omega}(\exists_{i<n}x_i)(\forall x)\bigvee\!\!\!\bigvee_{i<n}(x = x_i)\,.$$

- Well-foundedness (of the ambiant poset) is $\mathscr{L}_{\omega_1\omega_1}$:

$$(\forall_{n<\omega}x_n)\bigvee\!\!\!\bigvee_{n<\omega}(x_{n+1} \not< x_n)\,.$$

- Torsion-freeness (of a group) is $\mathscr{L}_{\omega_1\omega}$:

$$\bigwedge\!\!\!\bigwedge_{0<n<\omega}(\forall x)(x^n = 1 \Rightarrow x = 1)\,.$$

From fresh to elementary

For any first-order language Σ, we denote by $\mathbf{Str}\,\Sigma$ the category of all Σ-structures with Σ-homomorphisms.

From fresh to elementary

For any first-order language Σ, we denote by $\mathbf{Str}\,\Sigma$ the category of all Σ-structures with Σ-homomorphisms.

Proposition (W 2019)

Let λ be an infinite regular cardinal and let Σ be a first-order language. Then every λ-fresh homomorphism $f\colon \boldsymbol{A} \to \boldsymbol{B}$ in $\mathbf{Str}\,\Sigma$ is an $\mathscr{L}_{\infty\lambda}$-elementary embedding.

From fresh to elementary

For any first-order language Σ, we denote by $\mathbf{Str}\,\Sigma$ the category of all Σ-structures with Σ-homomorphisms.

Proposition (W 2019)

Let λ be an infinite regular cardinal and let Σ be a first-order language. Then every λ-fresh homomorphism $f\colon \boldsymbol{A} \to \boldsymbol{B}$ in $\mathbf{Str}\,\Sigma$ is an $\mathscr{L}_{\infty\lambda}$-elementary embedding.

- This says that $\boldsymbol{A} \models \varphi(\vec{a})$ iff $\boldsymbol{B} \models \varphi(f\vec{a})$, whenever φ is an $\mathscr{L}_{\infty\lambda}$ sentence with parameters $\vec{a}$ from $\boldsymbol{A}$.

From fresh to elementary

For any first-order language Σ, we denote by $\mathbf{Str}\,\Sigma$ the category of all Σ-structures with Σ-homomorphisms.

Proposition (W 2019)

Let λ be an infinite regular cardinal and let Σ be a first-order language. Then every λ-fresh homomorphism $f\colon \boldsymbol{A} \to \boldsymbol{B}$ in $\mathbf{Str}\,\Sigma$ is an $\mathscr{L}_{\infty\lambda}$-elementary embedding.

- This says that $\boldsymbol{A} \models \varphi(\vec{a})$ iff $\boldsymbol{B} \models \varphi(f\vec{a})$, whenever φ is an $\mathscr{L}_{\infty\lambda}$ sentence with parameters $\vec{a}$ from $\boldsymbol{A}$. This is proved by induction on the complexity of φ.

From fresh to elementary

For any first-order language Σ, we denote by $\mathbf{Str}\,\Sigma$ the category of all Σ-structures with Σ-homomorphisms.

Proposition (W 2019)

Let λ be an infinite regular cardinal and let Σ be a first-order language. Then every λ-fresh homomorphism $f \colon \boldsymbol{A} \to \boldsymbol{B}$ in $\mathbf{Str}\,\Sigma$ is an $\mathscr{L}_{\infty\lambda}$-elementary embedding.

- This says that $\boldsymbol{A} \models \varphi(\vec{a})$ iff $\boldsymbol{B} \models \varphi(f\vec{a})$, whenever φ is an $\mathscr{L}_{\infty\lambda}$ sentence with parameters $\vec{a}$ from $\boldsymbol{A}$. This is proved by induction on the complexity of φ.
- The case where φ is atomic is not completely trivial, and already follows from the λ-purity of f. Thus f is an embedding.

From functor to freshness

The following says that functors from $\mathfrak{P}_{\mathrm{inj}}(\Omega)$ to a category $\mathcal{C}$ create lots of fresh morphisms in $\mathcal{C}$.

Proposition (W 2019)

Let λ be an infinite regular cardinal, let $\mathcal{C}$ be a category, let Ω be a set, and let $\Gamma\colon \mathfrak{P}_{\mathrm{inj}}(\Omega) \to \mathcal{C}$ be a λ-continuous functor.

From functor to freshness

The following says that functors from $\mathfrak{P}_{\mathrm{inj}}(\Omega)$ to a category $\mathcal{C}$ create lots of fresh morphisms in $\mathcal{C}$.

Proposition (W 2019)

Let λ be an infinite regular cardinal, let $\mathcal{C}$ be a category, let Ω be a set, and let $\Gamma \colon \mathfrak{P}_{\mathrm{inj}}(\Omega) \to \mathcal{C}$ be a λ-continuous functor. Then for every $f \colon X \rightarrowtail Y$ in $\mathfrak{P}_{\mathrm{inj}}(\Omega)$ with $\operatorname{card} X \geq \lambda$, the morphism $\Gamma(f) \colon \Gamma(X) \to \Gamma(Y)$ is λ-fresh.

From functor to freshness

The following says that functors from $\mathfrak{P}_{\mathrm{inj}}(\Omega)$ to a category $\mathcal{C}$ create lots of fresh morphisms in $\mathcal{C}$.

Proposition (W 2019)

Let λ be an infinite regular cardinal, let $\mathcal{C}$ be a category, let Ω be a set, and let $\Gamma\colon \mathfrak{P}_{\mathrm{inj}}(\Omega) \to \mathcal{C}$ be a λ-continuous functor. Then for every $f\colon X \rightarrowtail Y$ in $\mathfrak{P}_{\mathrm{inj}}(\Omega)$ with $\operatorname{card} X \geq \lambda$, the morphism $\Gamma(f)\colon \Gamma(X) \to \Gamma(Y)$ is λ-fresh. In particular, if $\mathcal{C} = \mathbf{Str}\,\Sigma$ for a first-order language Σ, then $\Gamma(f)$ is an $\mathscr{L}_{\infty\lambda}$-elementary embedding.

From functor to freshness

The following says that functors from $\mathfrak{P}_{\mathrm{inj}}(\Omega)$ to a category $\mathcal{C}$ create lots of fresh morphisms in $\mathcal{C}$.

Proposition (W 2019)

Let λ be an infinite regular cardinal, let $\mathcal{C}$ be a category, let Ω be a set, and let $\Gamma\colon \mathfrak{P}_{\mathrm{inj}}(\Omega) \to \mathcal{C}$ be a λ-continuous functor. Then for every $f\colon X \rightarrowtail Y$ in $\mathfrak{P}_{\mathrm{inj}}(\Omega)$ with $\operatorname{card} X \geq \lambda$, the morphism $\Gamma(f)\colon \Gamma(X) \to \Gamma(Y)$ is λ-fresh. In particular, if $\mathcal{C} = \mathbf{Str}\,\Sigma$ for a first-order language Σ, then $\Gamma(f)$ is an $\mathscr{L}_{\infty\lambda}$-elementary embedding.

- λ-continuous means that Γ preserves λ-directed colimits.

From functor to freshness

The following says that functors from $\mathfrak{P}_{\mathrm{inj}}(\Omega)$ to a category $\mathcal{C}$ create lots of fresh morphisms in $\mathcal{C}$.

Proposition (W 2019)

Let λ be an infinite regular cardinal, let $\mathcal{C}$ be a category, let Ω be a set, and let $\Gamma\colon \mathfrak{P}_{\mathrm{inj}}(\Omega) \to \mathcal{C}$ be a λ-continuous functor. Then for every $f\colon X \rightarrowtail Y$ in $\mathfrak{P}_{\mathrm{inj}}(\Omega)$ with $\operatorname{card} X \geq \lambda$, the morphism $\Gamma(f)\colon \Gamma(X) \to \Gamma(Y)$ is λ-fresh. In particular, if $\mathcal{C} = \mathbf{Str}\,\Sigma$ for a first-order language Σ, then $\Gamma(f)$ is an $\mathscr{L}_{\infty\lambda}$-elementary embedding.

- λ-continuous means that Γ preserves λ-directed colimits. That is, from every λ-directed union $X = \bigcup_{i \in I} X_i$ we get a colimit $\Gamma(X) = \varinjlim_{i \in I} \Gamma(X_i)$.

From functor to freshness

The following says that functors from $\mathfrak{P}_{\mathrm{inj}}(\Omega)$ to a category $\mathcal{C}$ create lots of fresh morphisms in $\mathcal{C}$.

Proposition (W 2019)

Let λ be an infinite regular cardinal, let $\mathcal{C}$ be a category, let Ω be a set, and let $\Gamma\colon \mathfrak{P}_{\mathrm{inj}}(\Omega) \to \mathcal{C}$ be a λ-continuous functor. Then for every $f\colon X \rightarrowtail Y$ in $\mathfrak{P}_{\mathrm{inj}}(\Omega)$ with $\operatorname{card} X \geq \lambda$, the morphism $\Gamma(f)\colon \Gamma(X) \to \Gamma(Y)$ is λ-fresh. In particular, if $\mathcal{C} = \mathbf{Str}\,\Sigma$ for a first-order language Σ, then $\Gamma(f)$ is an $\mathscr{L}_{\infty\lambda}$-elementary embedding.

- λ-continuous means that Γ preserves λ-directed colimits. That is, from every λ-directed union $X = \bigcup_{i\in I} X_i$ we get a colimit $\Gamma(X) = \varinjlim_{i\in I} \Gamma(X_i)$.
- This result is a particular case of a more general preservation result of freshness under functors.

Anti-elementarity

Definition (W 2019)

A class $\mathcal{C}$ of objects in a category $\mathcal{S}$ is anti-elementary in $\mathcal{S}$ if there are arbitrarily large pairs $\lambda < \kappa$ of cardinals, with λ regular, and λ-continuous functors $\Gamma \colon \mathfrak{P}_{\mathrm{inj}}(\kappa) \to \mathcal{S}$ such that $\Gamma(\lambda) \in \mathcal{C}$ and $\Gamma(\kappa) \notin \mathcal{C}$.

Anti-elementarity

Definition (W 2019)

A class $\mathcal{C}$ of objects in a category $\mathcal{S}$ is anti-elementary in $\mathcal{S}$ if there are arbitrarily large pairs $\lambda < \kappa$ of cardinals, with λ regular, and λ-continuous functors $\Gamma\colon \mathfrak{P}_{\mathrm{inj}}(\kappa) \to \mathcal{S}$ such that $\Gamma(\lambda) \in \mathcal{C}$ and $\Gamma(\kappa) \notin \mathcal{C}$.

- By the previous result, the canonical morphism $e_\lambda^\kappa\colon \Gamma(\lambda) \to \Gamma(\kappa)$ is λ-fresh.

Anti-elementarity

Definition (W 2019)

A class $\mathcal{C}$ of objects in a category $\mathcal{S}$ is anti-elementary in $\mathcal{S}$ if there are arbitrarily large pairs $\lambda < \kappa$ of cardinals, with λ regular, and λ-continuous functors $\Gamma \colon \mathfrak{P}_{\mathrm{inj}}(\kappa) \to \mathcal{S}$ such that $\Gamma(\lambda) \in \mathcal{C}$ and $\Gamma(\kappa) \notin \mathcal{C}$.

- By the previous result, the canonical morphism $e_\lambda^\kappa \colon \Gamma(\lambda) \to \Gamma(\kappa)$ is λ-fresh. Thus, if $\mathcal{S} = \mathbf{Str}\,\Sigma$ for some first-order language Σ, then e_λ^κ is an $\mathscr{L}_{\infty\lambda}$-elementary embedding.

Anti-elementarity

Definition (W 2019)

A class $\mathcal{C}$ of objects in a category $\mathcal{S}$ is anti-elementary in $\mathcal{S}$ if there are arbitrarily large pairs $\lambda < \kappa$ of cardinals, with λ regular, and λ-continuous functors $\Gamma\colon \mathfrak{P}_{\mathrm{inj}}(\kappa) \to \mathcal{S}$ such that $\Gamma(\lambda) \in \mathcal{C}$ and $\Gamma(\kappa) \notin \mathcal{C}$.

- By the previous result, the canonical morphism $e_\lambda^\kappa\colon \Gamma(\lambda) \to \Gamma(\kappa)$ is λ-fresh. Thus, if $\mathcal{S} = \mathbf{Str}\,\Sigma$ for some first-order language Σ, then e_λ^κ is an $\mathscr{L}_{\infty\lambda}$-elementary embedding.
- In particular, $\mathcal{C}$ is not the class of all models of any class of $\mathscr{L}_{\infty\lambda}$ sentences.

Where does Γ come from?

- Typically, $\mathcal{S} = \mathbf{Str}\,\Sigma$ and $\mathcal{C}$ is the range of a functor $\Phi \colon \mathcal{A} \to \mathcal{S}$:

Where does Γ come from?

- Typically, $\mathcal{S} = \mathbf{Str}\,\Sigma$ and $\mathcal{C}$ is the range of a functor $\Phi\colon \mathcal{A} \to \mathcal{S}$: $R \mapsto$ finitely generated ideals of R (rings), $R \mapsto$ nonstable K-theory of R (rings), $G \mapsto$ Stone dual of the spectrum of G (lattice-ordered groups), and so on.

Where does Γ come from?

- Typically, $\mathcal{S} = \mathbf{Str}\,\Sigma$ and $\mathcal{C}$ is the range of a functor $\Phi\colon \mathcal{A} \to \mathcal{S}$: $R \mapsto$ finitely generated ideals of R (rings), $R \mapsto$ nonstable K-theory of R (rings), $G \mapsto$ Stone dual of the spectrum of G (lattice-ordered groups), and so on.
- The main difficulty is the construction of the functor Γ. It relies on the existence of a "Φ-commutative diagram" $\vec{A}$ from $\mathcal{A}$, indexed by (usually) a lattice P, such that $\Phi\vec{A} \ncong \Phi\vec{X}$ for any commutative diagram $\vec{X}$ from $\mathcal{A}$.

Where does Γ come from?

- Typically, $\mathcal{S} = \mathbf{Str}\,\Sigma$ and $\mathcal{C}$ is the range of a functor $\Phi\colon \mathcal{A} \to \mathcal{S}$: $R \mapsto$ finitely generated ideals of R (rings), $R \mapsto$ nonstable K-theory of R (rings), $G \mapsto$ Stone dual of the spectrum of G (lattice-ordered groups), and so on.
- The main difficulty is the construction of the functor Γ. It relies on the existence of a "Φ-commutative diagram" $\vec{A}$ from $\mathcal{A}$, indexed by (usually) a lattice P, such that $\Phi\vec{A} \ncong \Phi\vec{X}$ for any commutative diagram $\vec{X}$ from $\mathcal{A}$.
- Using infinite combinatorial properties of P, a certain "lifter" $\partial\colon P\langle\kappa\rangle \to P$ is constructed (usually $\kappa \geq \lambda^{+n}$, where $n = \dim P - 1$), then a "P-scaled Boolean algebra" $\mathbf{F}(P\langle\kappa\rangle)$, and then a "condensate" $\mathbf{F}(P\langle\kappa\rangle) \otimes_{\Phi}^{\lambda} \vec{A}$.

Where does Γ come from?

- Typically, $\mathcal{S} = \mathbf{Str}\,\Sigma$ and $\mathcal{C}$ is the range of a functor $\Phi\colon \mathcal{A} \to \mathcal{S}$: $R \mapsto$ finitely generated ideals of R (rings), $R \mapsto$ nonstable K-theory of R (rings), $G \mapsto$ Stone dual of the spectrum of G (lattice-ordered groups), and so on.
- The main difficulty is the construction of the functor Γ. It relies on the existence of a "Φ-commutative diagram" $\vec{A}$ from $\mathcal{A}$, indexed by (usually) a lattice P, such that $\Phi\vec{A} \not\cong \Phi\vec{X}$ for any commutative diagram $\vec{X}$ from $\mathcal{A}$.
- Using infinite combinatorial properties of P, a certain "lifter" $\partial\colon P\langle\kappa\rangle \to P$ is constructed (usually $\kappa \geq \lambda^{+n}$, where $n = \dim P - 1$), then a "P-scaled Boolean algebra" $\mathbf{F}(P\langle\kappa\rangle)$, and then a "condensate" $\mathbf{F}(P\langle\kappa\rangle) \otimes_{\Phi}^{\lambda} \vec{A}$.
- The functor Γ is given by $\Gamma(X) = \mathbf{F}(P\langle X\rangle) \otimes_{\Phi}^{\lambda} \vec{A}$.

Thanks for your attention!