Intractability for images of certain functors

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Main references

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- We present a method enabling to verify that a given class $\{\boldsymbol{M} \mid \varphi(\boldsymbol{M})\}$ cannot be "described" in certain ways.

An example

- A ring consists of a set R, binary operations $+\colon R \times R \to R$, $(x, y) \mapsto x + y$, $\cdot\colon R \times R \to R$, $(x, y) \mapsto x \cdot y$, and constants $0, 1 \in R$, subjected to certain rules (e.g., $x \cdot 1 = 1 \cdot x = x$; $(R, +, 0)$ is an abelian group; $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$; etc.).

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Describe all posets of the form $(\operatorname{Id} R, \subseteq)$.

In that particular case, this will lead to an intractability result.

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- The assignment $R \mapsto \operatorname{Id} R$, from rings to posets, can be extended to *homomorphisms*.

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- For such a map, we can define a map $\operatorname{Id} f\colon \operatorname{Id} R \to \operatorname{Id} S$, $X \mapsto$ ideal generated by $f(X)$. This map is order-preserving (in fact it preserves arbitrary ideal sums).
- We say that the assignment Id is a functor: defined on objects, extended to morphisms, natural rules ($\operatorname{Id}(f \circ g) = (\operatorname{Id} f) \circ (\operatorname{Id} g)$, etc.).

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- In order to improve legibility, use abbreviations.
- For example, $(\forall t)\Big(\big(t \leq x \text{ and } t \leq y\big) \Leftrightarrow t \leq z\Big)$ (a subformula of (Meet)) is often denoted $z = x \wedge y$.

An attempt at a description (cont'd)

- Similarly, there is a sentence saying that any two ideals X, Y have a least upper bound $X \vee Y$ (*here, the ideal generated by* $X \cup Y$, *usually denoted* $X + Y$), namely

$$(\forall x)(\forall y)(\exists z)(\forall t)\Big(\big(x \leq t \text{ and } y \leq t\big) \Leftrightarrow z \leq t\Big)\,. \quad \text{(Join)}$$

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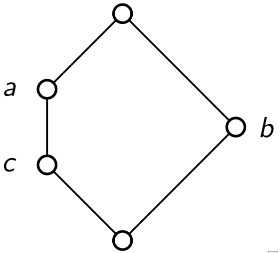
- Although the following poset satisfies both (Meet) and (Join) (it is a lattice), it does not appear as any $(\mathrm{Id}\, R, \subseteq)$.



Continuing the attempt (2)

- Reason for this: the modular law for ideal lattices of rings, $X \supseteq Z \Rightarrow X \cap (Y + Z) = (X \cap Y) + Z$, expressed by the first-order sentence

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- More complicated first-order sentences come up (e.g., the Arguesian law).

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- $\operatorname{Id} R$ and $\operatorname{Id}_c R$ can be obtained from each other: in that sense, describing one is describing the other.

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- Terms of a vocabulary $\mathbb{V}$ are (formal) compositions of operation symbols of $\mathbb{V}$. Atomic formulas have the form $s = t$ or $R(t_1, \dots, t_n)$, for terms s, t, t_i and n-ary relation symbols R.

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- For formulas φ and ψ of $\mathbb{V}$, their disjunction $\varphi \vee \psi$, their conjunction $\varphi \wedge \psi$, and the negation $\neg\varphi$ are also formulas.

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- A $\mathbb{v}$-structure is a nonempty set M, together with subsets $R^{\boldsymbol{M}} \subseteq M^n$ for $R \in \mathbb{v}_{\mathrm{rel}}$ and $\mathrm{ar}(R) = n$, and maps $f^{\boldsymbol{M}} \colon M^n \to M$ for $f \in \mathbb{v}_{\mathrm{ope}}$ and $\mathrm{ar}(f) = n$. Notation: $\boldsymbol{M} \in \mathbf{Str}(\mathbb{v})$.

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- Satisfaction, of a formula with parameters (free variable assignment) in a model $\boldsymbol{M}$, is defined by induction of the complexity of the formula: for example, $\boldsymbol{M} \models (\exists x)\varphi(x, \vec{a})$ means that there exists $b \in M$ such that $\boldsymbol{M} \models \varphi(b, \vec{a})$.

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- For example, a semigroup $\boldsymbol{M} = (M, \cdot)$ is commutative iff $\boldsymbol{M} \models (\forall x, y)(x \cdot y = y \cdot x)$.

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- For infinite cardinal numbers $\kappa \geq \lambda$, let $\mathscr{L}_{\kappa\lambda}(\mathbb{V})$ be the set of "infinitary formulas" of $\mathbb{V}$, defined in a similar way as first-order formulas, except that:
  1. The arities, of symbols in $\mathbb{V}$, may be ordinals $< \lambda$ (Example: Banach spaces, with $\lambda = \omega_1$);
  2. Iterated disjunctions $\bigvee_{i \in I} \varphi_i$ and conjunctions $\bigwedge_{i \in I} \varphi_i$, with card $I < \kappa$ and the φ_i have $< \lambda$ free variables altogether, are allowed;
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    3. Quantifications $\exists_{i \in I} x_i$ and $\forall_{i \in I} x_i$, with card $I < \lambda$, are allowed.
- Hence, $\mathscr{L}_{\omega\omega}(\mathbb{v})$ is the set of (ordinary) first-order formulas of $\mathbb{v}$.

Examples of infinitary sentences

- Finiteness can be expressed by a single $\mathscr{L}_{\omega_1\omega}$ sentence:

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A little background in category theory

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- Formally, categories are classes of objects related by arrows ("morphisms"). Invertible arrows are isomorphisms. Isomorphic objects are "the same".
- Formally, a category $\mathcal{S}$ consists of two disjoint classes $\mathrm{Ob}\,\mathcal{S}$ class $\mathrm{Ob}\,\mathcal{S}$ (the "objects" of $\mathcal{S}$), $\mathrm{Mor}\,\mathcal{S}$ (the "arrows" of $\mathcal{S}$), such that every arrow f is assigned two objects $\mathbf{d}(f)$ (the "domain" of f) and $\mathbf{r}(f)$ (the "range" of f) — in notation $f\colon \mathbf{d}(f) \to \mathbf{r}(f)$ — together with "identities" id_A (for $A \in \mathrm{Ob}\,\mathcal{S}$) and a partial binary "composition" operation $(f,g) \mapsto f \circ g$ on $\mathrm{Mor}\,\mathcal{S}$, with natural rules (e.g., $f \circ (g \circ h) = (f \circ g) \circ h$ whenever one side is defined, $f \circ \mathrm{id}_A = f$ whenever $f\colon A \to B$, etc.).

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- The class **Set** of all sets, with all maps, is a category.
- For any set Ω, we will consider later the category $[\Omega]^{\mathrm{inj}}$ of all subsets of Ω with one-to-one maps $f\colon X \rightarrowtail Y$ (where $X, Y \subseteq \Omega$) as arrows; it is a small category.

Functors, colimits

- A functor $\Phi\colon \mathcal{P} \to \mathcal{S}$, between categories $\mathcal{P}$ and $\mathcal{S}$, sends objects to objects and arrows to arrows, with natural rules (i.e., $\Phi(\mathrm{id}_A) = \mathrm{id}_{\Phi(A)}$, $\Phi(f \circ g) = \Phi(f) \circ \Phi(g)$).

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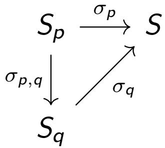
- It may happen that the diagram above has a colimit

$$(S, \sigma_p \mid p \in P) = \varinjlim \vec{S}.$$



λ-directed colimits, λ-continuous functors

- If, in the above, λ is an infinite regular cardinal and P is a λ-directed poset (i.e., every λ-small subset of P has an upper bound), we say that the colimit $S = \varinjlim \vec{S}$ is λ-directed.

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A categorical statement implying elementarity

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Proposition (W 2019)

Let λ be an infinite regular cardinal, let $\mathbb{V}$ be a first-order language, let Ω be a set, and let $\Gamma\colon [\Omega]^{\mathrm{inj}} \to \mathbf{Str}(\mathbb{V})$ be a λ-continuous functor. Then for every $f\colon X \rightarrowtail Y$ in $[\Omega]^{\mathrm{inj}}$ with $\operatorname{card} X \geq \lambda$, $\Gamma(f)$ is an $\mathscr{L}_{\infty\lambda}$-elementary embedding from $\Gamma(X)$ into $\Gamma(Y)$.

Anti-elementarity

Definition

A class $\mathcal{C}$ of objects, in a category $\mathcal{S}$, is anti-elementary if there are arbitrarily large cardinals $\lambda < \kappa$ with λ-continuous functors $\Gamma\colon [\kappa]^{\mathrm{inj}} \to \mathcal{S}$ such that $\Gamma(\lambda) \in \mathcal{C}$ and $\Gamma(\kappa) \notin \mathcal{C}$.

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- We shall outline a method making it possible to establish anti-elementarity for many classes. Those classes will always be images of functors (for a functor $\Phi\colon \mathcal{A} \to \mathcal{B}$, $\operatorname{im}\Phi \overset{\mathrm{def}}{=} \{B \mid (\exists A)(B \cong \Phi(A))\}$).

A few useful categories

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- $\mathbf{CMon} \stackrel{\text{def}}{=}$ category of all commutative monoids with monoid homomorphisms.

Functors for which the method works

Theorem (W 2019)

The images of the following functors are all anti-elementary:

1. $\mathrm{Cs_c} \colon \mathcal{G} \to \mathbf{DLat}_0$, $G \mapsto$ lattice of all order-convex ℓ-subgroups of the ℓ-group G; for any class $\mathcal{G}$ of ℓ-groups containing all Archimedean ones.
2. $\mathrm{Id_c} \colon \mathcal{R} \to \mathbf{SLat}_0$, $R \mapsto$ semilattice of all finitely generated two-sided ideals of R, for many classes $\mathcal{R}$ of rings, including all von Neumann regular rings and all rings.
3. $\mathrm{V} \colon \mathcal{R} \to \mathbf{CMon}$, $R \mapsto$ nonstable K_0-theory $\mathrm{V}(R)$ of R, for many classes $\mathcal{R}$ of rings, including all von Neumann regular rings and all C*-algebras of real rank zero.

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Under quite general conditions, the above implies that the image of Φ is anti-elementary.

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- Easy part of that description:

$$P\langle U\rangle \overset{\mathrm{def}}{=} \left\{ (a,x) \mid a \in P\,,\ x\colon X \to U\,,\ X \text{ finite}\,,\ a = \bigvee X \right\}$$

with $(a,x) \leq (b,y)$ iff $a \leq b$ and y extends x, and additional map $\partial\colon P\langle U\rangle \to P$, $(a,x) \mapsto a$.

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- It has order-dimension 3, thus one can take $\kappa = \aleph_{\alpha+2}$.

The diagrams $\vec{S}$ and $\vec{R}_{\mathbb{k}}$

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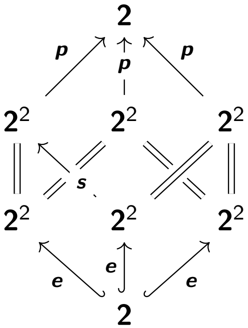

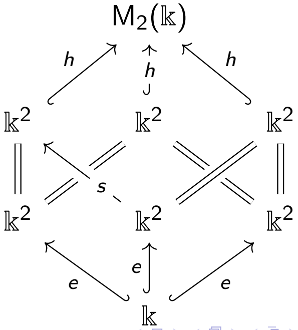

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- There is no commutative diagram $\vec{R}$ of rings such that $\mathrm{Id}_{\mathrm{c}}(\vec{R}) \cong \vec{S}$ (origin: late nineties, cf. W 2014; a bit more needs to be proved).

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- A closer look shows that $\mathrm{Id_c}(\mathbf{Ring})$ is not co-PC. This extends to all cases (nonstable K-theory, ℓ-groups...) considered above.