Spectrum problems for structures arising from lattices and rings

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- Is there an intrinsic characterization of the topological spaces of the form $\operatorname{Spec} A$?

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- $\operatorname{Spec} A$ is a spectral space, for every commutative unital ring A (well known and easy).

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- On the ring side, just consider unital ring homomorphisms.
- On the spectral space side, consider surjective spectral maps. For spectral spaces X and Y, a map $f : X \to Y$ is spectral if $f^{-1}[V] \in \mathring{\mathcal{K}}(X)$ whenever $V \in \mathring{\mathcal{K}}(Y)$.

The spectrum of a bounded distributive lattice

- A subset I in a bounded distributive lattice D is an ideal of D if $0 \in I$, $(\{x, y\} \subseteq I \Rightarrow x \vee y \in I)$, and $(\{x, y\} \cap I \neq \varnothing \Rightarrow x \wedge y \in I)$. An ideal I is prime if $I \neq D$ and $(x \wedge y \in I \Rightarrow \{x, y\} \cap I \neq \varnothing)$.

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- It is well known that the spectrum of any bounded distributive lattice is a spectral space.

The functors underlying Stone duality

- For bounded distributive lattices D and E and a $0,1$-lattice homomorphism $f\colon D \to E$, the map $\operatorname{Spec} f\colon \operatorname{Spec} E \to \operatorname{Spec} D,\ Q \mapsto f^{-1}[Q]$ is spectral.

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Note that in Hochster's Theorem's case, we do not obtain a duality (a ring is not determined by its spectrum).

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- In the case of bounded distributive lattices, we obtain a duality. In the case of commutative unital rings, we do not.
- Further algebraic structures also afford a concept of spectrum.

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- An order-unit of G is an element $e \in G^+$ such that $G = \langle e \rangle$.

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- It turns out that more is true!

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Not every completely normal spectral space is an ℓ-spectrum.

ℓ-spectra of Abelian ℓ-groups again

Theorem (Keimel 1971)

The ℓ-spectrum of any Abelian ℓ-group with unit is a completely normal spectral space.

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Delzell and Madden's example is not second countable (i.e., no countable basis of the topology): in fact, it has $\operatorname{card} \mathring{\mathcal{K}}(X) = \aleph_1$.

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- Since G has an order-unit, $\operatorname{Id}_c G$ is a bounded distributive lattice.
- Thus we must prove that every countable completely normal bounded distributive lattice D is $\cong \operatorname{Id}_c G$ for some Abelian ℓ-group G with unit.

Very rough outline of the proof of the countable case (cont'd)

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For bounded distributive lattices A and B, a $0,1$-lattice homomorphism $f\colon A \to B$ is closed if whenever $a_0, a_1 \in A$ and $b \in B$, if $f(a_0) \leq f(a_1) \vee b$, then there exists $x \in A$ such that $a_0 \leq a_1 \vee x$ and $f(x) \leq b$.

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Very rough outline of the proof of the countable case (further cont'd)

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- The subset $\operatorname{Op}^-(\mathcal{H}) \underset{\text{def}}{=} \operatorname{Op}(\mathcal{H}) \setminus \{\mathbb{E}\}$ is a sublattice of $\operatorname{Op}(\mathcal{H})$.

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- A crucial observation is that each $\mathrm{Op}(\mathcal{H})$ is a Heyting subalgebra of the Heyting algebra of all open subsets of $\mathbb{E}$.

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Difficult, involves categorical model-theoretical tools, developed in 2011 with Pierre Gillibert, called condensates.

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- It turns out that $\operatorname{Spec}_{\mathrm{r}} A$ is a completely normal spectral space, for any commutative unital ring A.

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- The countability assumption on $\mathbb{k}$ cannot be dispensed with (as witnessed by the lattice of all finite unions of relatively open, rational intervals of $[0,1]$).

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Theorem (Baro 2010)

Let $\Bbbk$ be a real-closed field, let $\mathbb{K}$ be a simplicial complex of $\Bbbk^d$, and let $S_1, \ldots, S_l$ be semi-algebraic subsets of $|\mathbb{K}|$. Then there exists a triangulation $(\mathbb{L}, \psi)$ of $(|\mathbb{K}|; S_1, \ldots, S_l)$ such that $\mathbb{L}$ is a subdivision of $\mathbb{K}$ and $\psi[S] = S$ for every open simplex S of $\mathbb{K}$.

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For any infinite cardinal λ, there is no $\mathscr{L}_{\infty,\lambda}$-characterization of the Stone duals of real spectra of commutative unital rings.

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It is known that every closed subspace of an ℓ-spectrum (resp., real spectrum) is an ℓ-spectrum (resp., real spectrum).

Theorem (W. 2017)

Not every spectral subspace of an ℓ-spectrum (resp., real spectrum) is an ℓ-spectrum (resp., real spectrum).

Problem (W. 2017)

Is a retract of an ℓ-spectrum also an ℓ-spectrum? Same question for real spectra.

Comparing spectra

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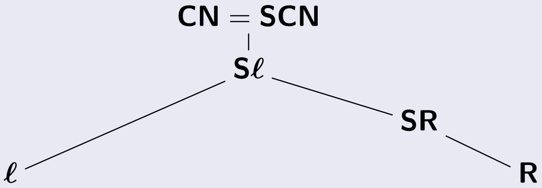
Theorem (W. 2017)

All containments and non-containments of the following picture are valid:



- All the separating counterexamples, intervening in the result above, have size $\aleph_1$, except for the counterexample witnessing $\mathbf{S}\ell \subsetneq \mathbf{CN}$, which has size $\aleph_2$.

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- Most of the examples constructed for the theorem above involve the construction of condensate (Gillibert and W. 2011), which turns diagram counterexamples to object counterexamples, with a jump of alephs corresponding to the order-dimension of the poset indexing the diagram (thus $\aleph_1$, $\aleph_2$, and so on).

A counterexample witnessing $\mathbf{R} \not\subseteq \ell$

- Knebusch and Scheiderer proved in 1989 that for any homomorphism $f \colon R \to S$ of commutative unital rings, the map $\mathrm{Spec_r}\, f \colon \mathrm{Spec_r}\, S \to \mathrm{Spec_r}\, R$ is convex, that is, whenever $Q_0 \subseteq Q_1$ in $\mathrm{Spec_r}\, S$, $P \in \mathrm{Spec_r}\, R$, and $f^{-1}Q_0 \subseteq P \subseteq f^{-1}Q_1$, there exists $Q \in \mathrm{Spec_r}\, S$ such that $Q_0 \subseteq Q \subseteq Q_1$ and $P = f^{-1}Q$.

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- Let K be any countable, non-Archimedean real-closed field, and set

$$A \underset{\mathrm{def}}{=} \{x \in K \mid (\exists n < \omega)(-n \cdot 1 \leq x \leq n \cdot 1\}\,.$$

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- The counterexample is the ring R of all almost constant families $(x_\xi \mid \xi < \omega_1) \in K^{\omega_1}$ such that $x_\infty \in A$: there is no Abelian ℓ-group G such that $\mathrm{Spec_r}\, R \cong \mathrm{Spec}_\ell\, G$.

A counterexample witnessing $\mathbf{SR} \not\subseteq \mathbf{R}$

- Start with a countable real-closed domain A with exactly three prime ideals $\{0\} \subsetneqq P_1 \subsetneqq P_2$. Then consider the ring E of all almost constant ω_1-indexed families of elements of A.

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- Start with a countable real-closed domain A with exactly three prime ideals $\{0\} \subsetneqq P_1 \subsetneqq P_2$. Then consider the ring E of all almost constant ω_1-indexed families of elements of A.
- Define $\varphi \colon \mathbf{4} \underset{\text{def}}{=} \{0, 1, 2, 3\} \twoheadrightarrow \mathbf{3} \underset{\text{def}}{=} \{0, 1, 2\}$ as the Stone dual of the (non-convex) map $\{1, 2\} \to \{1, 2, 3\}$, $1 \mapsto 1$, $2 \mapsto 3$. Hence $\varphi(0) = 0$, $\varphi(1) = \varphi(2) = 1$, $\varphi(3) = 2$.

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- The lattice $\mathrm{Cond}(\varphi,\omega_1) \underset{\text{def}}{=} \{(x,y) \in \mathbf{4} \times \mathbf{3}^{\omega_1} \mid y_\xi = \varphi(x)$ for all but finitely many $\xi\}$ is not the dual space of any real spectrum (because of Knebusch and Scheiderer's result).

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- However, $\mathrm{Cond}(\varphi, \omega_1)$ is a homomorphic image of the dual space of the real spectrum of E.

A counterexample witnessing $\ell \not\subseteq \mathbf{SR}$

- For any chain Λ, denote by $\mathbb{Z}\langle\Lambda\rangle$ the lexicographical power of $\mathbb{Z}$ by Λ: hence $\alpha < \beta$ in Λ implies that $n\alpha < \beta$ in $\mathbb{Z}\langle\Lambda\rangle$ for every integer n.

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- The counterexample is the lexicographical product $G \underset{\text{def}}{=} \mathbb{Z}\langle\omega_1^{\mathrm{op}}\rangle \times_{\text{lex}} F$:
- $\mathrm{Spec}_\ell\, G$ cannot be embedded, as a spectral subspace, into the real spectrum of any commutative unital ring.

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- Consider the variety $\mathcal{V}$, in the similarity type $(0, 1, \vee, \wedge, \smallsetminus)$, whose identities are those of bounded distributive lattices, together with the additional identities

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- It works because of Kuratowski's Free Set Theorem.

Thanks for your attention!