Representations of graph monoids by regular rings

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- $\mathrm{M}_\infty(R) := \varinjlim_{n<\omega} \mathrm{M}_n(R)$ is also a regular ring (without unit).

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- For R regular, the situation is far more complicated. . .

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- The monoid of all isomorphism types of Boolean algebras, with $[A] + [B] := [A \times B]$ (*it is also a proper class*).

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... and even still open in the finite case!

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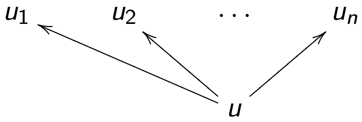

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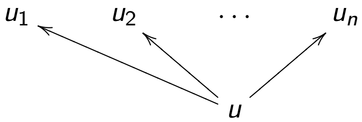

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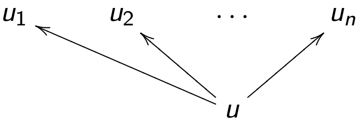

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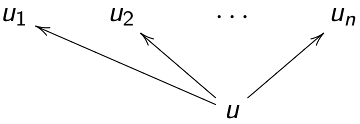

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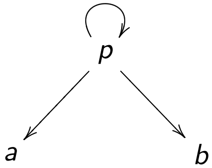

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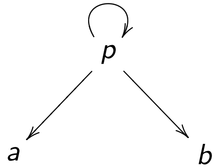

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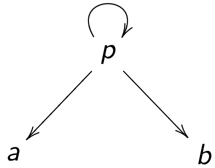

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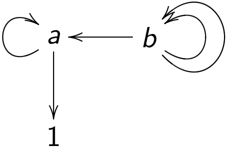

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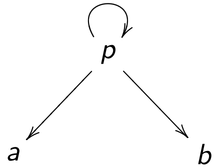

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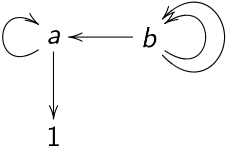

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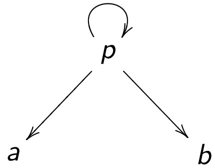

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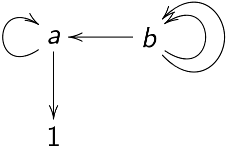

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$$a = a + 1\,, \quad b = 2b + a\,.$$

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The latter cannot be said for all the instances of the representation problem.

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- M is primitive, if it is an antisymmetric refinement monoid generated by its primes.

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- An element $p \in M$ with $p \not\leq 0$ is prime, if $p = x + y$ implies that either $p = x$ or $p = y$.
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- $\mathrm{Prime}(M) := \{p \in M \mid p \text{ is prime}\}$.
- M is primitive, if it is an antisymmetric refinement monoid generated by its primes.
- Equivalently, M is defined by generators and relations of the form $p_j = p_i + p_j$ $(\forall (i,j) \in \Gamma)$.

Theorem (Ara, Perera, W 2008)

A finitely generated primitive monoid M is a graph monoid iff it is a retract of a graph monoid, iff for each $p \in \mathrm{Prime}(M)$,

$$\{q \in \mathrm{Prime}_{\mathrm{free}}(M) \mid p \text{ covers } q \text{ in } \mathrm{Prime}(M)\}$$

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If we want, in addition, the quiver to be finite: we need the set of all free primes to be an upper subset of M.

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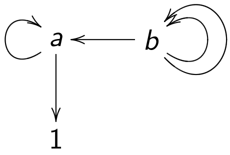
It is a retract of the graph monoid of the following quiver:



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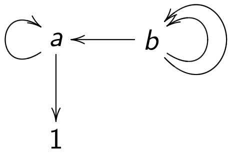

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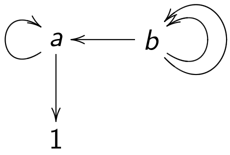

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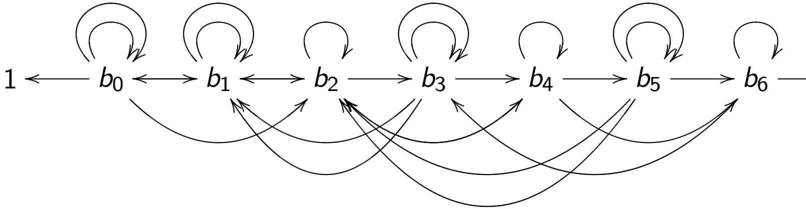

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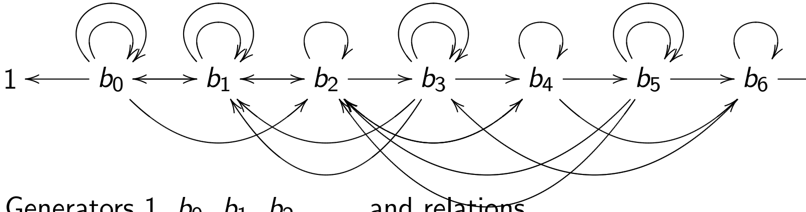

Generators 1, b_0, b_1, b_2, . . . , and relations

$$b_0 = 2b_0 + b_1 + b_2 + 1\,;$$
$$b_1 = b_0 + 2b_1 + b_2\,;$$
$$b_2 = b_2 + b_1 + b_3 + b_4\,;$$
$$b_3 = 2b_3 + 2b_1 + b_4\,;$$
$$\cdots \quad \cdots$$