Hidden identities

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Anatoly Alexeevich Karatsuba (1937–2008)

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By using the "divide and conquer" method, this enables to multiply large numbers much faster. For example, with two numbers of $2^{10} = 1{,}024$ digits, $3^{10} = 59{,}048$ multiplications (Karatsuba multiplication) instead of $2^{20} = 1{,}048{,}576$ (classical multiplication). There are even faster multiplication algorithms (e.g., FFT), but not so easy to implement.

Extending the validity range of remarkable identities

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Basic examples

Remarkable identities for matrices

Identities in rings

Other structures

Back to remarkable identities:

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How about more general objects?

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For this example, $(a+b)^2 \neq a^2 + 2ab + b^2$. Similarly, $(a-b)^2 \neq a^2 - 2ab + b^2$ and $(a+b)(a-b) \neq a^2 - b^2$.

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- Hence the problem boils down to $ab \neq ba$.

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- For $a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ (previous example), we obtain $ab = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $ba = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

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- The argument above is valid in any ring.

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(x\cdot y)\cdot z &= x\cdot(y\cdot z) && \text{(associativity of } \cdot\text{)}\,; \\
x\cdot(y+z) &= (x\cdot y)+(x\cdot z) && \text{(left distributivity)}\,; \\
(x+y)\cdot z &= (x\cdot z)+(y\cdot z) && \text{(right distributivity)}\,; \\
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The identity $x \cdot y = y \cdot x$ (commutativity of $\cdot$) defines commutative rings.

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$$\begin{aligned} & abcd - bacd - abdc + badc - acbd + cabd \\ & + acdb - cadb + adbc - dabc - adcb + dacb \\ & + cdab - cdba - dcab + dcba - bdac + bdca \\ & + dbac - dbca + bcad - bcda - cbad + cbda \\ & = 0\,. \end{aligned}$$

An identity for all $n \times n$ matrices:

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$$\sum_{\sigma \in \mathfrak{S}_{2n}} \operatorname{sgn}(\sigma) a_{\sigma(1)} \cdots a_{\sigma(2n)} = 0 \,,$$

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(here, $\operatorname{sgn}(\sigma)$ denotes the "signature" of the permutation σ),

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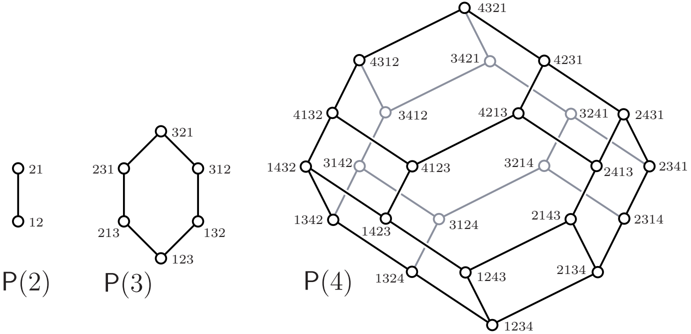
- Remark: If an identity holds in all real matrix rings (of arbitrary dimension), then it holds in all rings. This implies that "there is no version of the Amitsur-Levitzki identity that holds for all n".

Permutation lattices (“permutohedra”) on 2, 3, and 4 letters



If n is the number of inversions of σ (height of σ in the permutohedron), then $\text{sgn}(\sigma) = 1$ if n is even, -1 if n is odd.

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- In fact, for every positive integer n, every ring satisfying the identity $x^n = x$ is commutative (difficult result!).

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$$\text{(lattices)} \begin{cases} (x \vee y) \vee z = x \vee (y \vee z)\,; \\ x \vee y = y \vee x\,; \\ x \vee x = x\,; \\ (x \wedge y) \wedge z = x \wedge (y \wedge z)\,; \\ x \wedge y = y \wedge x\,; \\ x \wedge x = x\,; \\ x \wedge (x \vee y) = x \vee (x \wedge y) = x\,. \end{cases}$$

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)\,;$$

$$x \wedge \neg x = 0\,; x \vee \neg x = 1\,.$$

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  - $X + Y = (X \cup Y) \setminus (X \cap Y)$ (symmetric difference).

Robbins algebras

- Identities:

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\begin{aligned}
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- The problem of the converse was stated by Herbert Robbins in 1933.

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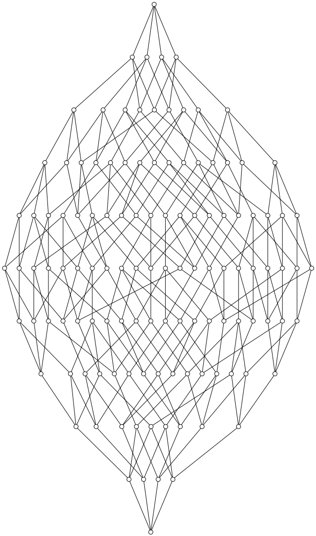
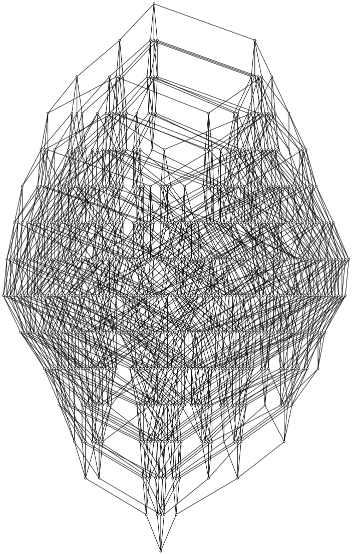
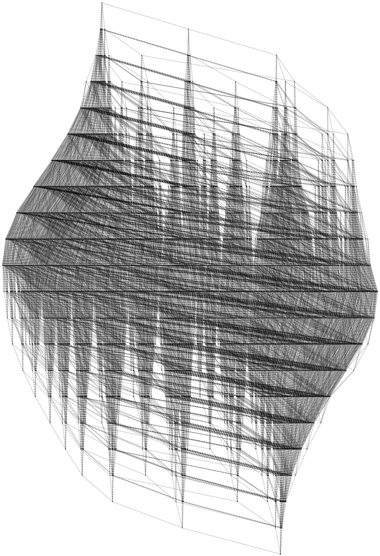
The permutohedron on 5 letters



The permutohedron on 6 letters



The permutohedron on 7 letters



Theorem (Santocanale and W. 2014)

There exists an identity, on the operations $\vee$ (join) and $\wedge$ (meet), that holds in all permutohedra $\mathrm{P}(n)$ (i.e., in all permutation lattices), but that does not hold in all (finite) lattices.

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- This identity could, in principle, be written explicitly. However, it would then fill a book.

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- Example of attempt: $x * y = x^3 + y^2$ is not identical to $y * x = x^2 + y^3$. Hence, the identity $x * y = y * x$ does not hold.

Thanks for your attention!