Projective classes as images of accessible functors

Friedrich Wehrung

Université de Caen
LMNO, CNRS UMR 6139
Département de Mathématiques
14032 Caen cedex

E-mail: friedrich.wehrung01@unicaen.fr
URL: http://wehrungf.users.lmno.cnrs.fr

March 2022
References


We would like to prove that certain “naturally defined” categories $\mathcal{C}$ of models (say of first-order theories) are “intractable”.
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- **Examples**: Posets of finitely generated ideals of rings, Ordered $K_0$ groups of unit-regular rings, Stone duals of spectra of abelian lattice-ordered groups,
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A way to define intractability is to state that $\mathcal{C}$ is not the class of models of any infinitary (not just first-order!) sentence (we’ll say elementary).
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A way to define intractability is to state that $\mathcal{C}$ is **not** the class of models of any infinitary (not just first-order!) sentence (we’ll say elementary).

Let’s suggest a stronger notion of intractability.
**v-structures**

- **Vocabulary:** $v = (v_{\text{ope}}, v_{\text{rel}}, ar)$ with $v_{\text{ope}} \cap v_{\text{rel}} = \emptyset$ and $ar: v_{\text{ope}} \cup v_{\text{rel}} \rightarrow$ ordinals (usually) with $0 \notin ar[v_{\text{rel}}]$. 

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- Elementary, projective
- Tuuri's Interpolation Theorem
- Karttunen's back-and-forth systems
**v-structures**

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- $ar(s) = 0 \iff s$ is a “constant”. 

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- Add to this a large enough set ("alphabet") of "variables".
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- **model for \( v \) (or v-structure)**: \( A = (A, s^A)_{s \in v_{\text{ope}} \cup v_{\text{rel}}} \), with the interpretations \( s^A \) defined the usual way.
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- **$\text{Str}(v)$** $\overset{\text{def}}{=} \text{category of all v-structures with v-homomorphisms (it is locally presentable).}$
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- **Terms**: closure of variables under all functions symbols.
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- $\text{Str}(v) \overset{\text{def}}{=} \text{category of all } v\text{-structures with } v\text{-homomorphisms}$ (it is **locally presentable**).
- **Terms**: closure of variables under all functions symbols.
- **atomic formulas**: $s = t$, for terms $s$ and $t$, or $R(t_\xi \mid \xi \in \text{ar}(R))$ where the $t_\xi$ are terms and $R \in v_{\text{rel}}$. 

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- Here $\kappa$ and $\lambda$ are “extended cardinals” ($\infty$ allowed) with $\omega \leq \lambda \leq \kappa \leq \infty$. 

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The languages $\mathcal{L}_{\kappa\lambda}$

- Here $\kappa$ and $\lambda$ are “extended cardinals” ($\infty$ allowed) with $\omega \leq \lambda \leq \kappa \leq \infty$.
- For any vocabulary $\mathbf{v}$, $\mathcal{L}_{\kappa\lambda}(\mathbf{v}) \overset{\text{def}}{=} \text{closure of all atomic } \mathbf{v}\text{-formulas under disjunctions of } < \kappa \text{ members } (\bigvee_{i \in I} E_i \text{ where card } I < \kappa), \text{ negation, and existential quantification over sets of less than } \lambda \text{ variables } ((\exists X)E \text{ with card } X < \lambda, \text{ or, in indexed form, } \exists \vec{x} E \text{ with card } I < \lambda).$
The languages \( \mathcal{L}_{\kappa\lambda} \)

- Here \( \kappa \) and \( \lambda \) are “extended cardinals” (\( \infty \) allowed) with \( \omega \leq \lambda \leq \kappa \leq \infty \).

- For any vocabulary \( v \), \( \mathcal{L}_{\kappa\lambda}(v) \defeq \) closure of all atomic \( v \)-formulas under disjunctions of \( < \kappa \) members (\( \bigvee_{i \in I} E_i \) where \( \text{card} \ I < \kappa \)), negation, and existential quantification over sets of less than \( \lambda \) variables (\( (\exists X)E \) with \( \text{card} \ X < \lambda \), or, in indexed form, \( \exists \vec{X} E \) with \( \text{card} \ I < \lambda \)).

- **Satisfaction** \( A \models E(\vec{a}) \) defined as usual (\( A \) is a \( v \)-structure, \( E \in \mathcal{L}_{\infty\infty}(v) \), \( \vec{a} \): free variables (\( E \rightarrow A \))).
The languages $\mathcal{L}_{\kappa\lambda}$

- Here $\kappa$ and $\lambda$ are “extended cardinals” ($\infty$ allowed) with $\omega \leq \lambda \leq \kappa \leq \infty$.

- For any vocabulary $\mathbf{v}$, $\mathcal{L}_{\kappa\lambda}(\mathbf{v}) \overset{\text{def}}{=} \text{closure of all atomic } \mathbf{v}\text{-formulas under disjunctions of } < \kappa \text{ members } (\bigvee_{i \in I} E_i \text{ where } \text{card } I < \kappa), \text{ negation, and existential quantification over sets of less than } \lambda \text{ variables } ((\exists X)E \text{ with } \text{card } X < \lambda, \text{ or, in indexed form, } \exists \vec{x} E \text{ with } \text{card } I < \lambda).$

- Satisfaction $A \models E(\vec{a})$ defined as usual ($A$ is a $\mathbf{v}$-structure, $E \in \mathcal{L}_{\infty\infty}(\mathbf{v})$, $\vec{a}$: free variables ($E \to A$).

- $\mathcal{L}_{\kappa\lambda}$-elementary class:
  $\mathcal{C} = \operatorname{Mod}_{\mathbf{v}}(E) \overset{\text{def}}{=} \{ A \in \operatorname{Str}(\mathbf{v}) \mid A \models E \}$ where $E$ is an $\mathcal{L}_{\kappa\lambda}(\mathbf{v})$-sentence.
(Relatively) projective classes

A class $\mathcal{C}$ of $\nu$-structures is

- relatively projective over $L_{\kappa \lambda}$ (abbrev. RPC($L_{\kappa \lambda}$)) if there are a unary predicate symbol $U$, a vocabulary $w \supseteq \nu \cup \{U\}$, and a sentence $E \in L_{\kappa \lambda}(w)$ such that $\mathcal{C} = \{U M \upharpoonright \nu | M \in \text{Mod}_w(E), U M \text{ closed under } \nu \text{ope}\}$. Hence $\text{PC}(L_{\kappa \lambda}) \subseteq \text{RPC}(L_{\kappa \lambda})$. Note that $\text{PC}(L_{\omega \omega}) \nsubseteq \text{RPC}(L_{\omega \omega})$ (even on finite structures).

Theorem (W 2021) Let $\lambda$ be an infinite cardinal. Then $\text{PC}(L_{\infty \lambda}) = \text{RPC}(L_{\infty \lambda})$ (in full generality; no restrictions on vocabularies). Moreover, if $\lambda$ is singular, then $\text{PC}(L_{\infty \lambda}) = \text{PC}(L_{\infty \lambda}^+)$. 

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A class \( \mathcal{C} \) of \( \nu \)-structures is

- **projective over** \( \mathcal{L}_{\kappa\lambda} \) (abbrev. \( \text{PC}(\mathcal{L}_{\kappa\lambda}) \)) if there are a vocabulary \( w \supseteq \nu \) and a sentence \( E \in \mathcal{L}_{\kappa\lambda}(w) \) such that
  \[
  \mathcal{C} = \{ M \upharpoonright \nu \mid M \in \text{Mod}_w(E) \}.
  \]
- **relatively projective over** \( \mathcal{L}_{\kappa\lambda} \) (abbrev. \( \text{RPC}(\mathcal{L}_{\kappa\lambda}) \)) if there are a unary predicate symbol \( U \), a vocabulary \( w \supseteq \nu \cup \{ U \} \), and a sentence \( E \in \mathcal{L}_{\kappa\lambda}(w) \) such that
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  \mathcal{C} = \{ U^M \upharpoonright \nu \mid M \in \text{Mod}_w(E), \ U^M \text{ closed under } \nu_{\text{ope}} \}.
  \]

Hence \( \text{PC}(\mathcal{L}_{\kappa\lambda}) \subseteq \text{RPC}(\mathcal{L}_{\kappa\lambda}) \). Note that \( \text{PC}(\mathcal{L}_{\omega\omega}) \not\subseteq \text{RPC}(\mathcal{L}_{\omega\omega}) \) (even on finite structures).

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Let \( \lambda \) be an infinite cardinal. Then \( \text{PC}(\mathcal{L}_{\infty\lambda}) = \text{RPC}(\mathcal{L}_{\infty\lambda}) \) (in full generality; no restrictions on vocabularies). Moreover, if \( \lambda \) is singular, then \( \text{PC}(\mathcal{L}_{\infty\lambda}) = \text{PC}(\mathcal{L}_{\infty\lambda}^+) \).
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A class $\mathcal{C}$ of $\mathbf{v}$-structures is

- **projective over** $\mathcal{L}_{\kappa, \lambda}$ (abbrev. $\text{PC}(\mathcal{L}_{\kappa, \lambda})$) if there are a vocabulary $\mathbf{w} \supseteq \mathbf{v}$ and a sentence $E \in \mathcal{L}_{\kappa, \lambda}(\mathbf{w})$ such that $\mathcal{C} = \{ \mathcal{M}|_{\mathbf{v}} \mid \mathcal{M} \in \text{Mod}_{\mathbf{w}}(E) \}$.

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Hence $\text{PC}(\mathcal{L}_{\kappa, \lambda}) \subseteq \text{RPC}(\mathcal{L}_{\kappa, \lambda})$. Note that $\text{PC}(\mathcal{L}_{\omega, \omega}) \not\subseteq \text{RPC}(\mathcal{L}_{\omega, \omega})$ (even on finite structures).
(Relatively) projective classes

A class \( \mathcal{C} \) of \( \mathbf{v} \)-structures is

- **projective over** \( \mathcal{L}_{k, \lambda} \) (abbrev. \( \text{PC}(\mathcal{L}_{k, \lambda}) \)) if there are a vocabulary \( \mathbf{w} \supseteq \mathbf{v} \) and a sentence \( E \in \mathcal{L}_{k, \lambda}(\mathbf{w}) \) such that
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- Hence \( \text{PC}(\mathcal{L}_{k, \lambda}) \subseteq \text{RPC}(\mathcal{L}_{k, \lambda}) \). Note that
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Let \( \lambda \) be an infinite cardinal. Then \( \text{PC}(\mathcal{L}_{\infty \lambda}) = \text{RPC}(\mathcal{L}_{\infty \lambda}) \) (in full generality; no restrictions on vocabularies). Moreover, if \( \lambda \) is singular, then \( \text{PC}(\mathcal{L}_{\infty \lambda}) = \text{PC}(\mathcal{L}_{\infty \lambda^+}) \).
Examples of “elementary” classes

- **Finiteness** (of the amiant universe) is $\mathcal{L}_{\omega_1\omega}$:

$$\bigwedge_{n<\omega} (\exists i<n x_i)(\forall x) \bigwedge_{i<n} (x = x_i).$$
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  \]

- **Torsion-freeness** (of a group) is $\mathcal{L}_{\omega_1\omega}$:
  \[
  \bigwedge_{0<n<\omega} (\forall x)(x^n = 1 \Rightarrow x = 1).
  \]
An example of RPC (that turns out to be PC)

\[ C \overset{\text{def}}{=} \{ M = (M, \cdot, 1) \text{ monoid} | (\exists G \text{ group})(M \hookrightarrow G) \} \text{ is, by definition, RPC}(\mathcal{L}_{\omega\omega}). \]
An example of RPC (that turns out to be PC)

- \( \mathcal{C} \overset{\text{def}}{=} \{ M = (M, \cdot, 1) \text{ monoid} \mid (\exists G \text{ group})(M \hookrightarrow G) \} \) is, by definition, RPC(\( \mathcal{L}_{\omega\omega} \)).

- Here \( v = (\cdot, 1) \), \( w = (\cdot, 1, U) \) for a unary predicate \( U \), the required \( E \) states that the given \( w \)-structure is a group (so “\( U^G \) is \( v \)-closed in \( G \)” means that \( U \) interprets a submonoid of \( G \)).
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- By Mal’cev’s work, \( \mathcal{C} = \{ M \mid (\forall n < \omega)(M \models E_n) \} \) for an effectively constructed sequence \((E_n \mid n < \omega)\) of quasi-identities over \( v \), not reducible to any finite subset.
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- \( \mathcal{C} \overset{\text{def}}{=} \{ M = (M, \cdot, 1) \text{ monoid} \mid (\exists G \text{ group})(M \hookrightarrow G) \} \) is, by definition, RPC(\( \mathcal{L}_{\omega\omega} \)).

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- Nonetheless, \( \mathcal{C} = \{ M \mid (\exists \text{ group structure } G \text{ on } M)(\exists f : M \hookrightarrow G) \} \) is PC(\( \mathcal{L}_{\omega\omega} \)).
Other examples

- For a unital ring $R$, $\text{Id}_c R \overset{\text{def}}{=} (\lor, 0)$-semilattice of all finitely generated two-sided ideals of $R$. Let $C \overset{\text{def}}{=} \{ \text{Id}_c R \mid R \text{ unital ring} \}$. 
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- For an Abelian $\ell$-group $G$, $\text{Id}_c G \overset{\text{def}}{=} \text{lattice of all principal } \ell\text{-ideals of } G$. Let $\mathcal{C} \overset{\text{def}}{=} \{ \text{Id}_c G \mid G \text{ Abelian } \ell\text{-group} \}$. 
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- For a commutative unital ring $A$, $\Phi(A) \overset{\text{def}}{=} \text{Stone dual of the real spectrum of } A$ (it is a bounded distributive lattice). Let $\mathcal{C} \overset{\text{def}}{=} \{ \Phi(A) \mid A \text{ commutative unital ring} \}$. 

All those classes are $\text{PC}(L_{\omega_1 \omega})$. Observe that they are all defined as images of functors. We will see that none of those classes is $\text{co-PC}(L_{\infty\infty})$ (i.e., complement of a $\text{PC}(L_{\infty\infty})$).
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- For an Abelian $\ell$-group $G$, $\text{Id}_c G \overset{\text{def}}{=} $ lattice of all principal $\ell$-ideals of $G$. Let $\mathcal{C} \overset{\text{def}}{=} \{ \text{Id}_c G \mid G \text{ Abelian } \ell\text{-group} \}$.
- For a commutative unital ring $A$, $\Phi(A) \overset{\text{def}}{=} $ Stone dual of the real spectrum of $A$ (it is a bounded distributive lattice). Let $\mathcal{C} \overset{\text{def}}{=} \{ \Phi(A) \mid A \text{ commutative unital ring} \}$.
- All those classes are $\text{PC}(\mathcal{L}_{\omega_1})$.
- Observe that they are all defined as images of functors.
- We will see that none of those classes is $\text{co-PC}(\mathcal{L}_{\infty})$ (i.e., complement of a $\text{PC}(\mathcal{L}_{\infty})$).
Let $\lambda$ be a regular cardinal.
Accessible categories and functors

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- A category \( S \) is \( \lambda \)-accessible if it has all \( \lambda \)-directed colimits and it has a \( \lambda \)-directed colimit-dense subset \( S^\dagger \), consisting of \( \lambda \)-presentable objects.
Let $\lambda$ be a regular cardinal.

- A category $\mathcal{S}$ is $\lambda$-accessible if it has all $\lambda$-directed colimits and it has a $\lambda$-directed colimit-dense subset $\mathcal{S}^\dagger$, consisting of $\lambda$-presentable objects.
- One can then take $\mathcal{S}^\dagger = \text{Pres}_\lambda \mathcal{S}$, “the” set of all $\lambda$-presentable objects in $\mathcal{S}$ (up to isomorphism).
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- A category $S$ is $\lambda$-accessible if it has all $\lambda$-directed colimits and it has a $\lambda$-directed colimit-dense subset $S^\dagger$, consisting of $\lambda$-presentable objects.
- One can then take $S^\dagger = \text{Pres}_\lambda S$, “the” set of all $\lambda$-presentable objects in $S$ (up to isomorphism).
- A functor $\Phi: S \to T$ is $\lambda$-continuous if it preserves $\lambda$-directed colimits. If $S$ and $T$ are both $\lambda$-accessible categories, we say that $\Phi$ is a $\lambda$-accessible functor.
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- A category $\mathcal{S}$ is $\lambda$-accessible if it has all $\lambda$-directed colimits and it has a $\lambda$-directed colimit-dense subset $\mathcal{S}^\dagger$, consisting of $\lambda$-presentable objects.
- One can then take $\mathcal{S}^\dagger = \operatorname{Pres}_{\lambda} \mathcal{S}$, “the” set of all $\lambda$-presentable objects in $\mathcal{S}$ (up to isomorphism).
- A functor $\Phi: \mathcal{S} \to \mathcal{T}$ is $\lambda$-continuous if it preserves $\lambda$-directed colimits. If $\mathcal{S}$ and $\mathcal{T}$ are both $\lambda$-accessible categories, we say that $\Phi$ is a $\lambda$-accessible functor.
- There are many examples: $\text{Str}(\mathbf{v})$, quasivarieties...
## Theorem (W 2021)

Let $\lambda$ be a regular cardinal, let $\mathbf{v}$ be a vocabulary such that $\mathbf{v}_{\text{ope}}$ is $\lambda$-ary, and let $\mathcal{C}$ be an $\text{RPC}(\mathcal{L}_{\infty \lambda})$ class of $\mathbf{v}$-structures. Then there are a $\lambda$-accessible category $\mathcal{S}$ and a $\lambda$-continuous functor $\Phi: \mathcal{S} \to \text{Str}(\mathbf{v})$, that can be taken faithful, with $\text{im } \Phi \overset{\text{def}}{=} \{ M \mid (\exists S \in \text{Ob } \mathcal{S})(M \cong \Phi(S)) \} = \mathcal{C}$. 

### Motivation

- Elementary, projective
- Tuuri's Interpolation Theorem
- Karttunen's back-and-forth systems

### Projective classes as images of accessible functors
PC versus accessible

Theorem (W 2021)

Let $\lambda$ be a regular cardinal, let $\mathbf{v}$ be a vocabulary such that $\mathbf{v}_{\text{ope}}$ is $\lambda$-ary, and let $\mathcal{C}$ be an RPC($\mathcal{L}_{\infty\lambda}$) class of $\mathbf{v}$-structures. Then there are a $\lambda$-accessible category $\mathcal{S}$ and a $\lambda$-continuous functor $\Phi: \mathcal{S} \to \text{Str}(\mathbf{v})$, that can be taken faithful, with $\text{im } \Phi \defeq \{ M \mid (\exists S \in \text{Ob } \mathcal{S})(M \cong \Phi(S)) \} = \mathcal{C}$.

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Projective classes as images of accessible functors

Motivation
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The assumptions that $\mathbf{v}_{\text{ope}}$, or $\mathbf{v}$, be $\lambda$-ary, cannot be dispensed with (counterexamples with idempotence, emptiness).
Infinitely deep languages

- **Idea**: extend $\mathcal{L}_{\kappa\lambda}$ in such a way that infinite alternations of quantifiers be enabled.
Infinitely deep languages

- **Idea:** extend $L_{\kappa\lambda}$ in such a way that infinite alternations of quantifiers be enabled.

- **Game formula** (of Gale-Stewart kind): $\exists \vec{x} \in E(\vec{x})$ is $(\forall x_0)(\exists x_1)(\forall x_2) \cdots E(x_0, x_1, x_2, \ldots)$. 

Satisfaction of an $M_{\kappa\lambda}(v)$-statement is expressed via the existence of a winning strategy in the associated game.
Infinitely deep languages

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- Can be interpreted via a game with two players, $\forall$ (who plays all $x_{2n}$) and $\exists$ (who plays all $x_{2n+1}$). Hence $\forall$ (resp., $\exists$) wins iff $E(x_0, x_1, x_2, \ldots)$ (resp., $\neg E(x_0, x_1, x_2, \ldots)$).

The game above has “clock” $\omega$.

The “infinitely deep language” $\mathcal{M}_{\kappa\lambda}(v)$ contains more general formulas than the $\exists\vec{x}E(\vec{x})$ above, now clocked by posets which are simultaneously trees and meet-semilattices, in which every node has $<\kappa$ upper covers and every branch has length a successor $<\lambda$. Satisfaction of an $\mathcal{M}_{\kappa\lambda}(v)$-statement is expressed via the existence of a winning strategy in the associated game.
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Infinitely deep languages

- **Idea**: extend $\mathcal{L}_{\kappa\lambda}$ in such a way that infinite alternations of quantifiers be enabled.

- **Game formula** (of Gale-Stewart kind): $\exists \vec{x} E(\vec{x})$ is $(\forall x_0)(\exists x_1)(\forall x_2) \cdots E(x_0, x_1, x_2, \ldots)$.

- Can be interpreted *via* a game with two players, ∀ (who plays all $x_{2n}$) and ∃ (who plays all $x_{2n+1}$). Hence ∀ (resp., ∃) wins iff $E(x_0, x_1, x_2, \ldots)$ (resp., $\neg E(x_0, x_1, x_2, \ldots)$).

- The game above has “clock” $\omega$.

- The “infinitely deep language” $\mathcal{M}_{\kappa\lambda}(\mathfrak{v})$ contains more general formulas than the $\exists \vec{x} E(\vec{x})$ above, now clocked by posets which are simultaneously trees and meet-semilattices, in which every node has $< \kappa$ upper covers and every branch has length a successor $< \lambda$.

- **Satisfaction** of an $\mathcal{M}_{\kappa\lambda}(\mathfrak{v})$-statement is expressed *via* the existence of a winning strategy in the associated game.
Tuuri’s Interpolation Theorem

**Theorem (Tuuri 1992)**

Let $\kappa$ be a regular cardinal, let $v$ be a $\kappa$-ary vocabulary, set $\lambda \overset{\text{def}}{=} \sup\{\kappa^\alpha \mid \alpha < \kappa\}$, and let $E$ and $F$ be $L_{\kappa+\kappa}(v)$-sentences such that the conjunction $E \land F$ has no $v$-model. Then there exists an $M_{\lambda+\lambda}(v)$-sentence $G$, with vocabulary the intersection of the vocabularies of $E$ and $F$, such that $\models (E \Rightarrow G)$ and $\models (G \Rightarrow \neg F)$.
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Here, $\neg G$ denotes the sentence obtained by interchanging $\lor$ and $\land$, $\exists$ and $\forall$, $A$ and $\neg A$ in the expression of $G$ by a tree-clocked game; it implies the usual negation $\neg G$ (which, however, is no longer an $M_{\lambda+\lambda}$-sentence).
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- Here, $\neg G$ denotes the sentence obtained by interchanging $\bigvee$ and $\bigwedge$, $\exists$ and $\forall$, $A$ and $\neg A$ in the expression of $G$ by a tree-clocked game; it implies the usual negation $\neg G$ (which, however, is no longer an $\mathcal{M}_{\lambda+\lambda}$-sentence).
- By a 1971 counterexample due to Malitz, $\mathcal{M}_{\lambda+\lambda}$ cannot be replaced by $\mathcal{L}_{\infty\infty}$ in the statement of Tuuri’s Theorem.
Projective and co-projective

Corollary

Let $\mathbf{v}$ be a vocabulary. Then for all classes $\mathcal{A}$ and $\mathcal{B}$ of $\mathbf{v}$-structures, if $\mathcal{A}$ is PC($L_{\infty\infty}$), $\mathcal{B}$ is co-PC($L_{\infty\infty}$), and $\mathcal{A} \subseteq \mathcal{B}$, then there exists an $M_{\infty\infty}(\mathbf{v})$-sentence $G$ such that $\mathcal{A} \subseteq \text{Mod}_v(G) \subseteq \mathcal{B}$. 
**Corollary**

Let $\mathbf{v}$ be a vocabulary. Then for all classes $\mathcal{A}$ and $\mathcal{B}$ of $\mathbf{v}$-structures, if $\mathcal{A}$ is $\text{PC}(\mathcal{L}_{\infty\infty})$, $\mathcal{B}$ is $\text{co-PC}(\mathcal{L}_{\infty\infty})$, and $\mathcal{A} \subseteq \mathcal{B}$, then there exists an $\mathcal{M}_{\infty\infty}(\mathbf{v})$-sentence $G$ such that $\mathcal{A} \subseteq \text{Mod}_{\mathbf{v}}(G) \subseteq \mathcal{B}$.

**Corollary**

In order to prove that a $\text{PC}(\mathcal{L}_{\infty\infty})$ class $\mathcal{C}$ of $\mathbf{v}$-structures is not $\text{co-PC}(\mathcal{L}_{\infty\infty})$, it suffices to prove that $\mathcal{C}$ is not $\mathcal{M}_{\infty\infty}(\mathbf{v})$-definable.
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Corollary

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But then, what is the advantage of $\mathcal{M}_{\infty\infty}$-definable over $\text{PC}(\mathcal{L}_{\infty\infty})$-definable or $\text{co-PC}(\mathcal{L}_{\infty\infty})$-definable?
That’s back-and-forth!

- There are several non-equivalent definitions of back-and-forth between models (extended to categorical model theory by Beke and Rosický in 2018).
That’s back-and-forth!

There are several non-equivalent definitions of back-and-forth between models (extended to categorical model theory by Beke and Rosický in 2018).

Definition (Karttunen 1979)

For a regular cardinal $\lambda$, a $\lambda$-back-and-forth system between models $M$ and $N$ over a vocabulary $v$ consists of a poset $(F, \sqsubseteq)$, together with a function $f \mapsto \bar{f}$ with domain $F$, such that each $\bar{f} : d(f) \cong r(f)$ with $d(f) \leq M$ and $r(f) \leq N$, and the following conditions hold:

1. $f \sqsubseteq g$ implies $\bar{f} \subseteq \bar{g}$;
2. $(F, \sqsubseteq)$ is $\lambda$-inductive;
3. whenever $f \in F$ and $x \in M$ (resp., $y \in N$), there is $g \in F$ such that $f \subseteq g$ and $x \in d(g)$ (resp., $y \in r(g)$).

We then write $M \leftrightarrow_\lambda N$. 
Let $\lambda$ be a regular cardinal and let $M$ and $N$ be structures over a vocabulary $v$. If $M \equiv_{\lambda} N$, then $M$ and $N$ satisfy the same $M_{\infty\lambda}(v)$-sentences.
Theorem (Karttunen 1979)

Let $\lambda$ be a regular cardinal and let $M$ and $N$ be structures over a vocabulary $v$. If $M \equiv^\lambda N$, then $M$ and $N$ satisfy the same $M_{\infty \lambda}(v)$-sentences.

- Extended by Karttunen to the even more general languages $N_{\infty \lambda}$. 
**Theorem (Karttunen 1979)**

Let $\lambda$ be a regular cardinal and let $M$ and $N$ be structures over a vocabulary $v$. If $M \leftrightarrow_\lambda N$, then $M$ and $N$ satisfy the same $M_{\infty\lambda}(v)$-sentences.

- Extended by Karttunen to the even more general languages $N_{\infty\lambda}$.
- The syntax for $N_{\infty\lambda}$ is far more complex than for $M_{\infty\lambda}$, the semantics are even trickier (not unique!).
Establishing intractability

- By the above,
Establishing intractability

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**Proposition**

In order to prove that a $PC(\mathcal{L}_{\infty\infty})$ class $\mathcal{C}$ of $v$-structures is not $co-PC(\mathcal{L}_{\infty\infty})$, it suffices to prove that it is not closed under $\leftrightarrow_{\lambda}$ for a suitable regular cardinal $\lambda$. 

Applies to earlier introduced examples $Id^c (unital rings)$, $Id^c (Abelian \ell$-groups), duals of real spectra of commutative unital rings, and many others: each of those classes fails to be closed under $\leftrightarrow_{\lambda}$. The real trouble is: find a back-and-forth system $F: M \leftrightarrow_{\lambda} N$ with $M \in \mathcal{C}$ and $N/\in \mathcal{C}$ (where $\mathcal{C}$ is the given class).
Establishing intractability

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**Proposition**

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- The real trouble is: find a back-and-forth system $\mathcal{F}: M \leftrightarrow_\lambda N$ with $M \in \mathcal{C}$ and $N \notin \mathcal{C}$ (where $\mathcal{C}$ is the given class).
Back-and-forth systems from continuous functors

- In many examples, such as $\Phi(\text{unital rings})$ and $\Phi(\text{Abelian } \ell\text{-groups})$ (where $\Phi = \text{Id}_c$), $\leftrightarrow_\lambda$ arises from some $\lambda$-continuous functor $\Gamma : [\kappa]^{\text{inj}} \to \mathcal{C}$ with $\kappa \geq \lambda$. 
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Finding $P$ and $\vec{S}$ is usually hard, very much connected to the algebraic and combinatorial data of the given problem.
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- In many examples, such as \( \Phi(\text{unital rings}) \) and \( \Phi(\text{Abelian } \ell\text{-groups}) \) (where \( \Phi = \text{Id}_c \)), \( \cong \lambda \) arises from some \( \lambda \)-continuous functor \( \Gamma: [\kappa]_{\text{inj}} \to C \) with \( \kappa \geq \lambda \). Here, \( [\kappa]_{\text{inj}} \) denotes the category of all subsets of \( \kappa \) with one-to-one functions. In both examples above, \( \kappa = \lambda^{++} \).

- It is often the case that for \( X \subseteq \kappa \) with \( \text{card } X < \lambda \), \( \Gamma(X) = \Phi(\prod(S_{|u|} \mid u \in X^{\subseteq P})) \) (a “condensate”), where:
  1. \( P \) is a suitable finite lattice (in both examples above, \( P = \{0, 1\}^3 \); also, this method provably fails for arbitrary finite bounded posets!);
  2. \( X^{\subseteq P} \equiv \bigcup \{X^D \mid D \subseteq P\} \);
  3. \( |u| \equiv \bigvee \text{dom } u \) whenever \( u \in X^{\subseteq P} \);
  4. \( \vec{S} \) is a non-commutative diagram, indexed by \( P \), such that, for the given functor \( \Phi \), the diagram \( \Phi(\vec{S}) \) is commutative.
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The diagram $\tilde{S}$ for $\text{Id}_c(\text{Abelian } \ell\text{-groups})$

$$\begin{align*}
A_{123}(a, a', b, c) \\
A_{12}(a, b) & \quad A_{13}(a', c) & \quad A_{23}(b, c) \\
A_{1}(a) \quad & A_{2}(b) \quad & A_{3}(c) \\
A_{\emptyset} = \{0\}
\end{align*}$$

$$0 \leq a \leq a' \leq 2a; \ b \geq 0; \ c \geq 0.$$ 

$A_1(a) \rightarrow A_{13}(a', c) \text{ via } a \mapsto a'$. 

Motivation
Elementary, projective
Tuuri's Interpolation Theorem
Karttunen's back-and-forth systems
A further example with Abelian $\ell$-groups

Denote by $\mathcal{A}$ the class of all Abelian $\ell$-groups, and by $\text{Id}_c \mathcal{A}$ the class of all isomorphic copies of $\text{Id}_c G$ where $G \in \mathcal{A}$. It is $\text{PC}(\mathcal{L}_{\omega_1 \omega})$, but, by the above, not $\text{co-PC}(\mathcal{L}_{\infty \infty})$. 

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- A bounded distributive lattice \( D \) satisfies Ploščica’s Condition if for every \( a \in D \) and every collection \( (m_i \mid i \in I) \) of maximal ideals of \( \downarrow a, \downarrow a/\bigcap_i m_i \) has cardinality \( \leq 2^{\text{card} I} \) (careful with definition of \( \downarrow a/J \)).
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**Theorem (Ploščica 2021)**

Every member of $\text{Id}_c \mathcal{A}$ satisfies Ploščica’s Condition.
A further example with Abelian $\ell$-groups

- Denote by $\mathcal{A}$ the class of all Abelian $\ell$-groups, and by $\text{Id}_c \mathcal{A}$ the class of all isomorphic copies of $\text{Id}_c G$ where $G \in \mathcal{A}$. It is $\text{PC}(\mathcal{L}_{\omega_1 \omega})$, but, by the above, not $\text{co-PC}(\mathcal{L}_{\infty \infty})$.
- A bounded distributive lattice $D$ satisfies Ploščica’s Condition if for every $a \in D$ and every collection $\langle m_i \mid i \in I \rangle$ of maximal ideals of $\downarrow a$, $\downarrow a / \bigcap_i m_i$ has cardinality $\leq 2^{\text{card} I}$ (careful with definition of $\downarrow a / J$).

Theorem (Ploščica 2021)

Every member of $\text{Id}_c \mathcal{A}$ satisfies Ploščica’s Condition.

Theorem (W 2022, under a fragment of GCH)

There exists a bounded distributive lattice, of cardinality $\aleph_4$, satisfying all known $\mathcal{L}_{\omega_1 \omega_1}$ properties of all members of $\text{Id}_c \mathcal{A}$ together with Ploščica’s Condition, but not in $\text{Id}_c \mathcal{A}$. 
Thanks for your attention!