Modular lattices and von Neumann regular rings

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A **projective geometry** is a structure \( (P, L, \epsilon) \), where both \( P \) ("points") and \( L \) ("lines") are sets and \( \epsilon \subseteq P \times L \).
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Background: projective geometries

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1. Every line contains at least two distinct points; 
2. Any two distinct points are contained in exactly one line; 
3. The Pasch Axiom (more detail later!).
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so write \(p \in \ell\) instead of \(p \in \ell\).
The Pasch Axiom

A triangle is a triple \((p, q, r)\) of distinct points, such that \(p \not\in (q \ r)\), \(q \not\in (p \ r)\), and \(r \not\in (p \ q)\).
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![Diagram of the Pasch Axiom](image)
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X \wedge Y \text{ (meet)} := X \cap Y , \\
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\]

The structure \((\text{Sub } P, \vee, \wedge)\) (the **subspace lattice** of \( P \)) is a lattice.
Modularity of Sub $P$

Lattice Theory

Lattice Theory is the study of all structures $(L, \lor, \land)$, where $L$ is a nonempty set and $\lor$ (resp., $\land$) is the join operation (resp., meet operation) with respect to a (necessarily unique) partial ordering of $L$. In particular, Sub $P$ is a lattice. It is, in fact, a very special sort of lattice.

Lemma

The lattice Sub $P$ is modular, that is, it satisfies the rule $x \geq z \Rightarrow x \land (y \lor z) = (x \land y) \lor z$ (the modular law).
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$$(x \lor z) \land (y \lor z) = (x \lor z) \land y \lor z,$$

$$(x \land y) \lor (x \land z) = x \land (y \lor (x \land z)).$$
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Setting \( x \equiv x \lor z \) (resp., \( z \equiv x \land z \)), we get two equivalent forms of the modular law, formulated as identities:
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Each of these identities (defining modularity) is called the modular identity. A lattice $L$ is modular if and only if it does not contain a (lattice-)copy of the lattice $N_5$ below:
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(x \lor z) \land (y \lor z) = ((x \lor z) \land y) \lor z, \\
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\]
The modular identity

Setting $x := x ∨ z$ (resp., $z := x ∧ z$), we get two equivalent forms of the modular law, formulated as identities:

\[
(x ∨ z) ∧ (y ∨ z) = ((x ∨ z) ∧ y) ∨ z,
\]
\[
(x ∧ y) ∨ (x ∧ z) = x ∧ (y ∨ (x ∧ z)).
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Projective subspace lattices = geomodular lattices

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A lattice is geomodular if and only if it is isomorphic to Sub $P$, for some projective geometry $P$. 
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Every geomodular lattice $L$ is complemented,
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Every geomodular lattice $L$ is complemented, that is, for each $x \in L$, there exists $y \in L$ such that $x \lor y = 1$ (largest element of $L$) and $x \land y = 0$ (smallest element of $L$). (Abbreviated $x \oplus y = 1$, and we say that $y$ is a complement of $x$.)
Desargues’ Rule

**Definition**

Two triangles \((a_0, a_1, a_2)\) and \((b_0, b_1, b_2)\) are **centrally perspective**, if

- \((a_i a_j) \neq (b_i b_j)\) for all \(i \neq j\), and
- for some point \(p\), all points \(a_i, b_i, p\) are collinear (i.e., on the same line).

We say that \((a_0, a_1, a_2)\) and \((b_0, b_1, b_2)\) are **axially perspective**, if

- the points \(c_0, c_1, c_2\) are collinear, where \((a_1 a_2) \cap (b_1 b_2) = \{c_0\}\) and cyclically.

We say that the projective geometry \(P\) is **Arguesian** (or satisfies Desargues’ Rule), if

- any two centrally perspective triangles are also axially perspective.
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Desargues’ Rule

Definition

Two triangles \((a_0, a_1, a_2)\) and \((b_0, b_1, b_2)\) are centrally perspective, if \((a_i a_j) \neq (b_i b_j)\) for all \(i \neq j\), and for some point \(p\), all points \(a_i, b_i, p\) are collinear (i.e., on the same line).

We say that \((a_0, a_1, a_2)\) and \((b_0, b_1, b_2)\) are axially perspective, if the points \(c_0, c_1,\) and \(c_2\) are collinear, where \((a_1 a_2) \cap (b_1 b_2) = \{c_0\}\) and cyclically.

We say that the projective geometry \(P\) is Arguesian (or satisfies Desargues’ Rule), if any two centrally perspective triangles are also axially perspective.
Illustrating Desargues’ Rule
Illustrating Desargues’ Rule
The Arguesian identity

Desargues’ identity (M. Schützenberger 1945, B. Jónsson 1953)
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### Desargues’ identity (M. Schützenberger 1945, B. Jónsson 1953)

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- (more general) Any lattice of permuting equivalence relations on a given set. (Note: ‘Arguesian’ is then not the end of the story...)
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Complemented modular lattice (CML):

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Von Neumann frames

Definition

Elements $a$, $b$ in a modular lattice $L$ with $0$ are perspective with axis $c$ (notation $a \sim c b$), if $a \oplus c = b \oplus c$.

Elements $a_0, \ldots, a_{n-1}$ are independent, if $a_k \land \bigvee_{i < k} a_i = 0$, for each $k < n$.

An $n$-frame is a system $((a_i | 0 \leq i < n), (c_i | 1 \leq i < n))$, where $((a_i | 0 \leq i < n)$ is independent and $a_0 \sim c_i a_i$ for $1 \leq i < n$.

The frame is — spanning, if $1 = \bigvee_{i < n} a_i$, — large, if every element of $L$ is a finite join of elements perspective to parts of $a_0$.

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Definition

A ring (associative, not necessarily unital) $R$ is regular (in von Neumann's sense), if it satisfies $(\forall x)(\exists y)(xyx = x)$.

Example: the endomorphism ring of a vector space (or even a semisimple module) is regular.

One can then prove that $L(R) := \{x \in R | x \in R\}$ is a sublattice of the lattice $\text{Id}_R$ of all right ideals of $R$; in particular, it is modular.

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Theorem (Von Neumann 1936, Fryer and Halperin 1954)

The lattice $L(R)$ is modular, and also sectionally complemented, the latter meaning that $(\forall x \leq y)(\exists z)(x \oplus z = y)$. In particular, $L(R)$ is complemented modular if (and only if) $R$ is unital. (For modular lattices, complemented $\iff$ sectionally complemented with unit.)

Definition

A lattice is coordinatizable, if it is isomorphic to $L(R)$, for some regular ring $R$. The easiest example of non-coordinatizable CML is $M_7$. 

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If a CML has a spanning $n$-frame, with $n \geq 4$, then it is coordinatizable.

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If a CML has a large 4-frame, or it is Arguesian and it has a large 3-frame, then it is coordinatizable.

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Von Neumann’s condition requires the lattice have a unit, while Jónsson’s does not. Nevertheless, Jónsson’s Coordinatization Theorem is stated for lattices with unit.

For sectionally complemented modular lattices without unit, Jónsson’s result extends to the countable case (B. Jónsson 1962) . . . but not to the general case (FW 2008, counterexample of cardinality $\aleph_1$).

The proof of the latter counterexample involves Banaschewski functions (first used in 1957, in the theory of totally ordered abelian groups), and larders (P. Gillibert and FW, 2008; a tool of categorical nature).
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Word problem for modular lattices

Theorem (C. Herrmann 1983)

The word problem for free modular lattices on four generators is recursively unsolvable. The corresponding statement with 'five' instead of 'four' was proved by R. Freese in 1980.

The free modular lattice on three generators is finite, with 28 elements (R. Dedekind 1900)—so one can’t go down to ‘three’.

Remark

The word problem for all lattices is solvable in polynomial time. The word problem for all distributive lattices is \textit{NP}-complete.
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If a lattice $L$ embeds into some CML, is this also the case for all homomorphic images of $L$?
Another problem...

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Let $R$ be a (unital) regular ring. Denote by $V(R)$ the commutative monoid of all isomorphism types of finitely generated projective right $R$-modules. Is $V(R)$ separative, that is, does it satisfy the following statement:

\[(\forall x, y)(2x = 2y = x + y \Rightarrow x = y)\] 

The problem above is also open for C*-algebras of real rank zero, and even for general (Warfield) exchange rings.
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