Modular lattices and von Neumann regular rings

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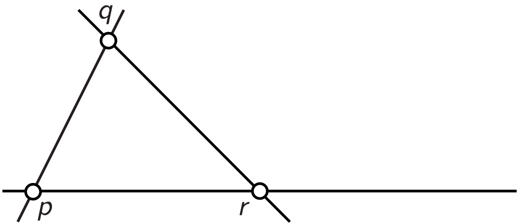

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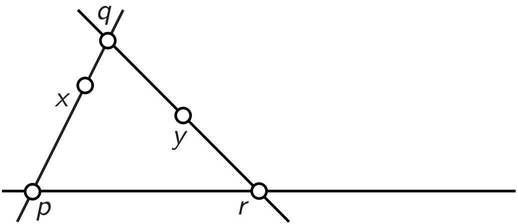

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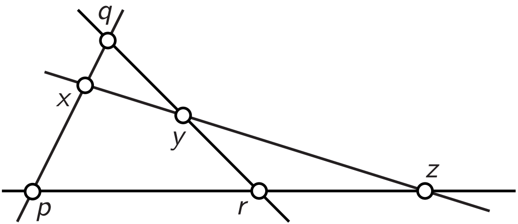

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The structure $(\operatorname{Sub} P, \vee, \wedge)$ (the subspace lattice of P) is a lattice.

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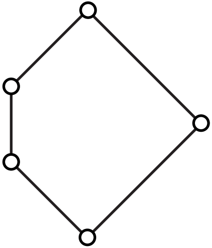

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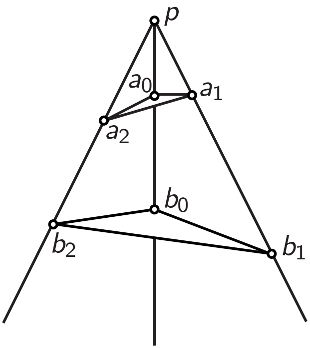
Illustrating Desargues' Rule



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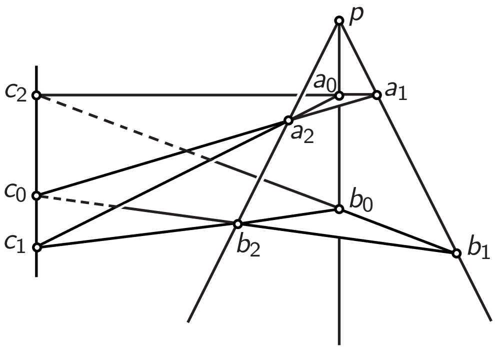

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- (*more general*) Any lattice of permuting equivalence relations on a given set. (*Note: 'Arguesian' is then not the end of the story...*)

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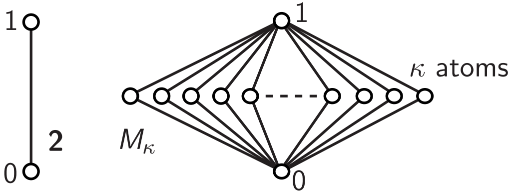

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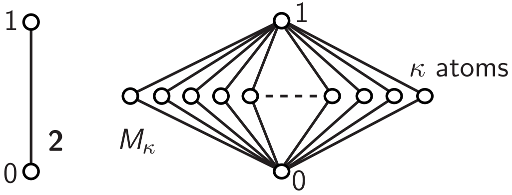

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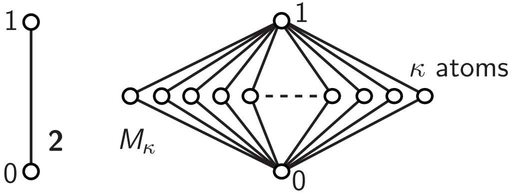

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(3) . . . *and the non-Arguesian projective planes*!

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The decomposition above is unique.

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A much more transparent proof of Jónsson's Coordinatization Theorem has recently been found by C. Herrmann.

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The proof of the latter counterexample involves Banaschewski functions (first used in 1957, in the theory of totally ordered abelian groups), and larders (P. Gillibert and FW, 2008; a tool of categorical nature).

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- The word problem for all lattices is solvable in polynomial time.

Word problem for modular lattices

Theorem (C. Herrmann 1983)

The word problem for free modular lattices on four generators is recursively unsolvable.

The corresponding statement with 'five' instead of 'four' was proved by R. Freese in 1980.
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- The word problem for all lattices is solvable in polynomial time.
- The word problem for all distributive lattices is **NP**-complete.

Open problems

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If a lattice L embeds into some CML, is this also the case for all homomorphic images of L?

Another problem. . .

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The problem above is also open for C*-algebras of real rank zero, and even for general (Warfield) exchange rings.

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