Spectrum problems for structures arising from lattices and rings

Friedrich Wehrung

Université de Caen
LMNO, CNRS UMR 6139
Département de Mathématiques
14032 Caen cedex
E-mail: friedrich.wehrung01@unicaen.fr
URL: http://wehrungf.users.lmno.cnrs.fr

SYSMICS Les Diablerets, August 25, 2018

The spectrum of a commutative, unital ring

- A proper ideal P in a commutative, unital ring A is prime if A/P is a domain. Equivalently, $xy \in P \Rightarrow (x \in P$ or $y \in P)$, for all $x, y \in A$.

The spectrum of a commutative, unital ring

- A proper ideal P in a commutative, unital ring A is prime if A/P is a domain. Equivalently, $xy \in P \Rightarrow (x \in P$ or $y \in P)$, for all $x, y \in A$.
- Endow the set $\operatorname{Spec} A \underset{\text{def}}{=} \{P \mid P \text{ is a prime ideal of } A\}$ with the topology whose closed sets are those of the form

The spectrum of a commutative, unital ring

- A proper ideal P in a commutative, unital ring A is prime if A/P is a domain. Equivalently, $xy \in P \Rightarrow (x \in P$ or $y \in P)$, for all $x, y \in A$.
- Endow the set $\operatorname{Spec} A \underset{\mathrm{def}}{=} \{P \mid P \text{ is a prime ideal of } A\}$ with the topology whose closed sets are those of the form

$$\operatorname{Spec}(A, X) \underset{\mathrm{def}}{=} \{P \in \operatorname{Spec} A \mid X \subseteq P\},$$

for $X \subseteq A$.

The spectrum of a commutative, unital ring

- A proper ideal P in a commutative, unital ring A is prime if A/P is a domain. Equivalently, $xy \in P \Rightarrow (x \in P$ or $y \in P)$, for all $x, y \in A$.
- Endow the set $\operatorname{Spec} A \underset{\text{def}}{=} \{P \mid P \text{ is a prime ideal of } A\}$ with the topology whose closed sets are those of the form

$$\operatorname{Spec}(A, X) \underset{\text{def}}{=} \{P \in \operatorname{Spec} A \mid X \subseteq P\},$$

for $X \subseteq A$.
- This is the so-called hull-kernel topology on $\operatorname{Spec} A$. The topological space thus obtained is the (Zariski) spectrum of A.

The spectrum of a commutative, unital ring

- A proper ideal P in a commutative, unital ring A is prime if A/P is a domain. Equivalently, $xy \in P \Rightarrow (x \in P$ or $y \in P)$, for all $x, y \in A$.
- Endow the set $\operatorname{Spec} A \underset{\text{def}}{=} \{P \mid P \text{ is a prime ideal of } A\}$ with the topology whose closed sets are those of the form

$$\operatorname{Spec}(A, X) \underset{\text{def}}{=} \{P \in \operatorname{Spec} A \mid X \subseteq P\},$$

  for $X \subseteq A$.
- This is the so-called hull-kernel topology on $\operatorname{Spec} A$. The topological space thus obtained is the (Zariski) spectrum of A.
- Is there an intrinsic characterization of the topological spaces of the form $\operatorname{Spec} A$?

Spectral spaces

- A nonempty closed set F in a topological space X is irreducible if $F = A \cup B$ implies that either $F = A$ or $F = B$, for all closed sets A and B.

Spectral spaces

- A nonempty closed set F in a topological space X is irreducible if $F = A \cup B$ implies that either $F = A$ or $F = B$, for all closed sets A and B.
- We say that X is sober if every irreducible closed set is $\overline{\{x\}}$ (the closure of $\{x\}$) for a unique $x \in X$.

Spectral spaces

- A nonempty closed set F in a topological space X is irreducible if $F = A \cup B$ implies that either $F = A$ or $F = B$, for all closed sets A and B.
- We say that X is sober if every irreducible closed set is $\overline{\{x\}}$ (the closure of $\{x\}$) for a unique $x \in X$.
- Set $\overset{\circ}{\mathcal{K}}(X) \underset{\text{def}}{=} \{U \subseteq X \mid U \text{ is open and compact}\}$.

Spectral spaces

- A nonempty closed set F in a topological space X is irreducible if $F = A \cup B$ implies that either $F = A$ or $F = B$, for all closed sets A and B.
- We say that X is sober if every irreducible closed set is $\overline{\{x\}}$ (the closure of $\{x\}$) for a unique $x \in X$.
- Set $\mathring{\mathcal{K}}(X) \underset{\text{def}}{=} \{U \subseteq X \mid U \text{ is open and compact}\}$.
- In general, $U, V \in \mathring{\mathcal{K}}(X) \Rightarrow U \cup V \in \mathring{\mathcal{K}}(X)$.

Spectral spaces

- A nonempty closed set F in a topological space X is irreducible if $F = A \cup B$ implies that either $F = A$ or $F = B$, for all closed sets A and B.
- We say that X is sober if every irreducible closed set is $\overline{\{x\}}$ (the closure of $\{x\}$) for a unique $x \in X$.
- Set $\overset{\circ}{\mathcal{K}}(X) \underset{\text{def}}{=} \{U \subseteq X \mid U \text{ is open and compact}\}$.
- In general, $U, V \in \overset{\circ}{\mathcal{K}}(X) \Rightarrow U \cup V \in \overset{\circ}{\mathcal{K}}(X)$.

  However, usually $U, V \in \overset{\circ}{\mathcal{K}}(X) \not\Rightarrow U \cap V \in \overset{\circ}{\mathcal{K}}(X)$.

Spectral spaces

- A nonempty closed set F in a topological space X is irreducible if $F = A \cup B$ implies that either $F = A$ or $F = B$, for all closed sets A and B.
- We say that X is sober if every irreducible closed set is $\overline{\{x\}}$ (the closure of $\{x\}$) for a unique $x \in X$.
- Set $\overset{\circ}{\mathcal{K}}(X) \underset{\text{def}}{=} \{U \subseteq X \mid U \text{ is open and compact}\}$.
- In general, $U, V \in \overset{\circ}{\mathcal{K}}(X) \Rightarrow U \cup V \in \overset{\circ}{\mathcal{K}}(X)$.

  However, usually $U, V \in \overset{\circ}{\mathcal{K}}(X) \not\Rightarrow U \cap V \in \overset{\circ}{\mathcal{K}}(X)$.
- We say that X is *spectral* if it is sober and $\overset{\circ}{\mathcal{K}}(X)$ is a basis of the topology of X, closed under finite intersection.

Spectral spaces

- A nonempty closed set F in a topological space X is irreducible if $F = A \cup B$ implies that either $F = A$ or $F = B$, for all closed sets A and B.
- We say that X is sober if every irreducible closed set is $\overline{\{x\}}$ (the closure of $\{x\}$) for a unique $x \in X$.
- Set $\overset{\circ}{\mathcal{K}}(X) \underset{\text{def}}{=} \{U \subseteq X \mid U \text{ is open and compact}\}$.
- In general, $U, V \in \overset{\circ}{\mathcal{K}}(X) \Rightarrow U \cup V \in \overset{\circ}{\mathcal{K}}(X)$.
  However, usually $U, V \in \overset{\circ}{\mathcal{K}}(X) \not\Rightarrow U \cap V \in \overset{\circ}{\mathcal{K}}(X)$.
- We say that X is *spectral* if it is sober and $\overset{\circ}{\mathcal{K}}(X)$ is a basis of the topology of X, closed under finite intersection. Taking the empty intersection then yields that X is compact (usually not Hausdorff).

Spectral spaces

- A nonempty closed set F in a topological space X is irreducible if $F = A \cup B$ implies that either $F = A$ or $F = B$, for all closed sets A and B.
- We say that X is sober if every irreducible closed set is $\overline{\{x\}}$ (the closure of $\{x\}$) for a unique $x \in X$.
- Set $\overset{\circ}{\mathcal{K}}(X) \underset{\text{def}}{=} \{U \subseteq X \mid U \text{ is open and compact}\}$.
- In general, $U, V \in \overset{\circ}{\mathcal{K}}(X) \Rightarrow U \cup V \in \overset{\circ}{\mathcal{K}}(X)$.
  However, usually $U, V \in \overset{\circ}{\mathcal{K}}(X) \not\Rightarrow U \cap V \in \overset{\circ}{\mathcal{K}}(X)$.
- We say that X is *spectral* if it is sober and $\overset{\circ}{\mathcal{K}}(X)$ is a basis of the topology of X, closed under finite intersection. Taking the empty intersection then yields that X is compact (usually not Hausdorff).
- Spec A is a spectral space, for every commutative unital ring A (well known and easy).

Hochster's Theorem

The converse of the above observation holds:

Theorem (Hochster 1969)

Every spectral space X is homeomorphic to $\operatorname{Spec} A$ for some commutative unital ring A.

Hochster's Theorem

The converse of the above observation holds:

Theorem (Hochster 1969)

Every spectral space X is homeomorphic to $\operatorname{Spec} A$ for some commutative unital ring A.

- Moreover, Hochster proves that the assignment $X \mapsto A$ can be made functorial.

Hochster's Theorem

The converse of the above observation holds:

Theorem (Hochster 1969)

Every spectral space X is homeomorphic to $\operatorname{Spec} A$ for some commutative unital ring A.

- Moreover, Hochster proves that the assignment $X \mapsto A$ can be made functorial.
- In order for that observation to make sense, the morphisms need to be specified.

Hochster's Theorem

The converse of the above observation holds:

Theorem (Hochster 1969)

Every spectral space X is homeomorphic to $\operatorname{Spec} A$ for some commutative unital ring A.

- Moreover, Hochster proves that the assignment $X \mapsto A$ can be made functorial.
- In order for that observation to make sense, the morphisms need to be specified.
- On the ring side, just consider unital ring homomorphisms.

Hochster's Theorem

The converse of the above observation holds:

Theorem (Hochster 1969)

Every spectral space X is homeomorphic to $\operatorname{Spec} A$ for some commutative unital ring A.

- Moreover, Hochster proves that the assignment $X \mapsto A$ can be made functorial.
- In order for that observation to make sense, the morphisms need to be specified.
- On the ring side, just consider unital ring homomorphisms.
- On the spectral space side, consider surjective spectral maps.

Hochster's Theorem

The converse of the above observation holds:

Theorem (Hochster 1969)

Every spectral space X is homeomorphic to $\operatorname{Spec} A$ for some commutative unital ring A.

- Moreover, Hochster proves that the assignment $X \mapsto A$ can be made functorial.
- In order for that observation to make sense, the morphisms need to be specified.
- On the ring side, just consider unital ring homomorphisms.
- On the spectral space side, consider surjective spectral maps. For spectral spaces X and Y, a map $f : X \to Y$ is spectral if $f^{-1}[V] \in \mathring{\mathcal{K}}(X)$ whenever $V \in \mathring{\mathcal{K}}(Y)$.

The spectrum of a bounded distributive lattice

- A subset I in a bounded distributive lattice D is an ideal of D if $0 \in I$, $(\{x, y\} \subseteq I \Rightarrow x \vee y \in I)$, and $(\{x, y\} \cap I \neq \varnothing \Rightarrow x \wedge y \in I)$. An ideal I is prime if $I \neq D$ and $(x \wedge y \in I \Rightarrow \{x, y\} \cap I \neq \varnothing)$.

The spectrum of a bounded distributive lattice

- A subset I in a bounded distributive lattice D is an ideal of D if $0 \in I$, $(\{x, y\} \subseteq I \Rightarrow x \vee y \in I)$, and $(\{x, y\} \cap I \neq \varnothing \Rightarrow x \wedge y \in I)$. An ideal I is prime if $I \neq D$ and $(x \wedge y \in I \Rightarrow \{x, y\} \cap I \neq \varnothing)$.
- For a bounded distributive lattice D, set $\operatorname{Spec} D \underset{\text{def}}{=} \{P \mid P \text{ is a prime ideal of } D\}$, endowed with the topology whose closed sets are the sets of the form

The spectrum of a bounded distributive lattice

- A subset I in a bounded distributive lattice D is an ideal of D if $0 \in I$, $(\{x, y\} \subseteq I \Rightarrow x \vee y \in I)$, and $(\{x, y\} \cap I \neq \varnothing \Rightarrow x \wedge y \in I)$. An ideal I is prime if $I \neq D$ and $(x \wedge y \in I \Rightarrow \{x, y\} \cap I \neq \varnothing)$.
- For a bounded distributive lattice D, set $\operatorname{Spec} D \underset{\text{def}}{=} \{P \mid P \text{ is a prime ideal of } D\}$, endowed with the topology whose closed sets are the sets of the form

$$\operatorname{Spec}(D, X) \underset{\text{def}}{=} \{P \in \operatorname{Spec} D \mid X \subseteq P\}, \quad \text{for } X \subseteq D,$$

and we call it the spectrum of D.

The spectrum of a bounded distributive lattice

- A subset I in a bounded distributive lattice D is an ideal of D if $0 \in I$, $(\{x, y\} \subseteq I \Rightarrow x \vee y \in I)$, and $(\{x, y\} \cap I \neq \varnothing \Rightarrow x \wedge y \in I)$. An ideal I is prime if $I \neq D$ and $(x \wedge y \in I \Rightarrow \{x, y\} \cap I \neq \varnothing)$.
- For a bounded distributive lattice D, set $\operatorname{Spec} D \underset{\text{def}}{=} \{P \mid P \text{ is a prime ideal of } D\}$, endowed with the topology whose closed sets are the sets of the form

$$\operatorname{Spec}(D, X) \underset{\text{def}}{=} \{P \in \operatorname{Spec} D \mid X \subseteq P\}\,, \quad \text{for } X \subseteq D\,,$$

  and we call it the spectrum of D.
- It is well known that the spectrum of any bounded distributive lattice is a spectral space.

The functors underlying Stone duality

- For bounded distributive lattices D and E and a $0, 1$-lattice homomorphism $f\colon D \to E$, the map $\operatorname{Spec} f\colon \operatorname{Spec} E \to \operatorname{Spec} D,\ Q \mapsto f^{-1}[Q]$ is spectral.

The functors underlying Stone duality

- For bounded distributive lattices D and E and a 0, 1-lattice homomorphism $f: D \to E$, the map $\operatorname{Spec} f: \operatorname{Spec} E \to \operatorname{Spec} D$, $Q \mapsto f^{-1}[Q]$ is spectral.
- For spectral spaces X and Y and a spectral map $\varphi: X \to Y$, the map $\mathring{\mathcal{K}}(\varphi): \mathring{\mathcal{K}}(Y) \to \mathring{\mathcal{K}}(X)$, $V \mapsto \varphi^{-1}[V]$ is a 0, 1-lattice homomorphism.

The functors underlying Stone duality

- For bounded distributive lattices D and E and a $0,1$-lattice homomorphism $f\colon D \to E$, the map $\operatorname{Spec} f\colon \operatorname{Spec} E \to \operatorname{Spec} D$, $Q \mapsto f^{-1}[Q]$ is spectral.
- For spectral spaces X and Y and a spectral map $\varphi\colon X \to Y$, the map $\mathring{\mathcal{K}}(\varphi)\colon \mathring{\mathcal{K}}(Y) \to \mathring{\mathcal{K}}(X)$, $V \mapsto \varphi^{-1}[V]$ is a $0,1$-lattice homomorphism.

Theorem (Stone 1938)

The pair $(\operatorname{Spec}, \mathring{\mathcal{K}})$ induces a (categorical) duality, between bounded distributive lattices with $0,1$-lattice homomorphisms and spectral spaces with spectral maps.

The functors underlying Stone duality

- For bounded distributive lattices D and E and a $0,1$-lattice homomorphism $f\colon D \to E$, the map $\operatorname{Spec} f\colon \operatorname{Spec} E \to \operatorname{Spec} D$, $Q \mapsto f^{-1}[Q]$ is spectral.
- For spectral spaces X and Y and a spectral map $\varphi\colon X \to Y$, the map $\mathring{\mathcal{K}}(\varphi)\colon \mathring{\mathcal{K}}(Y) \to \mathring{\mathcal{K}}(X)$, $V \mapsto \varphi^{-1}[V]$ is a $0,1$-lattice homomorphism.

Theorem (Stone 1938)

The pair $(\operatorname{Spec}, \mathring{\mathcal{K}})$ induces a (categorical) duality, between bounded distributive lattices with $0,1$-lattice homomorphisms and spectral spaces with spectral maps.

Note that in Hochster's Theorem's case, we do not obtain a duality (a ring is not determined by its spectrum).

Further spectra?

- To summarize: spectral spaces are the same as spectra of commutative unital rings, and also spectra of bounded distributive lattices.

Further spectra?

- To summarize: spectral spaces are the same as spectra of commutative unital rings, and also spectra of bounded distributive lattices.
- In the case of bounded distributive lattices, we obtain a duality.

Further spectra?

- To summarize: spectral spaces are the same as spectra of commutative unital rings, and also spectra of bounded distributive lattices.
- In the case of bounded distributive lattices, we obtain a duality. In the case of commutative unital rings, we do not.

Further spectra?

- To summarize: spectral spaces are the same as spectra of commutative unital rings, and also spectra of bounded distributive lattices.
- In the case of bounded distributive lattices, we obtain a duality. In the case of commutative unital rings, we do not.
- Further algebraic structures also afford a concept of spectrum.

ℓ-ideals of an Abelian ℓ-group

- An ℓ-group is a group endowed with a lattice ordering $\leq$, such that $x \leq y$ implies both $xz \leq yz$ and $zx \leq zy$.

ℓ-ideals of an Abelian ℓ-group

- An ℓ-group is a group endowed with a lattice ordering $\leq$, such that $x \leq y$ implies both $xz \leq yz$ and $zx \leq zy$.
- The underlying lattice of an ℓ-group is necessarily distributive.

ℓ-ideals of an Abelian ℓ-group

- An ℓ-group is a group endowed with a lattice ordering $\leq$, such that $x \leq y$ implies both $xz \leq yz$ and $zx \leq zy$.
- The underlying lattice of an ℓ-group is necessarily distributive.
- Our ℓ-groups will be Abelian ($xy = yx$), thus we will denote them additively ($x + y = y + x$, $G^+ \underset{\text{def}}{=} \{x \in G \mid x \geq 0\}$, $|x| \underset{\text{def}}{=} x \vee (-x)$).

ℓ-ideals of an Abelian ℓ-group

- An ℓ-group is a group endowed with a lattice ordering $\leq$, such that $x \leq y$ implies both $xz \leq yz$ and $zx \leq zy$.
- The underlying lattice of an ℓ-group is necessarily distributive.
- Our ℓ-groups will be Abelian ($xy = yx$), thus we will denote them additively ($x + y = y + x$, $G^+ \underset{\text{def}}{=} \{x \in G \mid x \geq 0\}$, $|x| \underset{\text{def}}{=} x \vee (-x)$).
- An additive subgroup of an Abelian ℓ-group G is an ℓ-ideal if it is both order-convex and closed under $x \mapsto |x|$.

ℓ-ideals of an Abelian ℓ-group

- An ℓ-group is a group endowed with a lattice ordering $\leq$, such that $x \leq y$ implies both $xz \leq yz$ and $zx \leq zy$.
- The underlying lattice of an ℓ-group is necessarily distributive.
- Our ℓ-groups will be Abelian ($xy = yx$), thus we will denote them additively ($x + y = y + x$, $G^+ \underset{\text{def}}{=} \{x \in G \mid x \geq 0\}$, $|x| \underset{\text{def}}{=} x \vee (-x)$).
- An additive subgroup of an Abelian ℓ-group G is an ℓ-ideal if it is both order-convex and closed under $x \mapsto |x|$.
- An ℓ-ideal I of G is

ℓ-ideals of an Abelian ℓ-group

- An ℓ-group is a group endowed with a lattice ordering $\leq$, such that $x \leq y$ implies both $xz \leq yz$ and $zx \leq zy$.
- The underlying lattice of an ℓ-group is necessarily distributive.
- Our ℓ-groups will be Abelian ($xy = yx$), thus we will denote them additively ($x + y = y + x$, $G^+ \underset{\text{def}}{=} \{x \in G \mid x \geq 0\}$, $|x| \underset{\text{def}}{=} x \vee (-x)$).
- An additive subgroup of an Abelian ℓ-group G is an ℓ-ideal if it is both order-convex and closed under $x \mapsto |x|$.
- An ℓ-ideal I of G is
  - prime if $I \neq G$ and $x \wedge y \in I \Rightarrow \{x, y\} \cap I \neq \varnothing$.

ℓ-ideals of an Abelian ℓ-group

- An ℓ-group is a group endowed with a lattice ordering $\leq$, such that $x \leq y$ implies both $xz \leq yz$ and $zx \leq zy$.
- The underlying lattice of an ℓ-group is necessarily distributive.
- Our ℓ-groups will be Abelian ($xy = yx$), thus we will denote them additively ($x + y = y + x$, $G^+ \underset{\text{def}}{=} \{x \in G \mid x \geq 0\}$, $|x| \underset{\text{def}}{=} x \vee (-x)$).
- An additive subgroup of an Abelian ℓ-group G is an ℓ-ideal if it is both order-convex and closed under $x \mapsto |x|$.
- An ℓ-ideal I of G is
  - prime if $I \neq G$ and $x \wedge y \in I \Rightarrow \{x, y\} \cap I \neq \varnothing$.
  - finitely generated (equivalently, principal) if $I = \langle a \rangle = \{x \in G \mid (\exists n)(|x| \leq na)\}$ for some $a \in G^+$.

ℓ-ideals of an Abelian ℓ-group

- An ℓ-group is a group endowed with a lattice ordering $\leq$, such that $x \leq y$ implies both $xz \leq yz$ and $zx \leq zy$.
- The underlying lattice of an ℓ-group is necessarily distributive.
- Our ℓ-groups will be Abelian ($xy = yx$), thus we will denote them additively ($x + y = y + x$, $G^+ \underset{\text{def}}{=} \{x \in G \mid x \geq 0\}$, $|x| \underset{\text{def}}{=} x \vee (-x)$).
- An additive subgroup of an Abelian ℓ-group G is an ℓ-ideal if it is both order-convex and closed under $x \mapsto |x|$.
- An ℓ-ideal I of G is
  - prime if $I \neq G$ and $x \wedge y \in I \Rightarrow \{x, y\} \cap I \neq \varnothing$.
  - finitely generated (equivalently, principal) if $I = \langle a \rangle = \{x \in G \mid (\exists n)(|x| \leq na)\}$ for some $a \in G^+$.
- An order-unit of G is an element $e \in G^+$ such that $G = \langle e \rangle$.

The ℓ-spectrum of an Abelian ℓ-group with unit

- For an Abelian ℓ-group G with (order-)unit, we set $\operatorname{Spec}_\ell G \underset{\text{def}}{=} \{P \mid P \text{ is a prime ideal of } G\}$, endowed with the topology whose closed sets are the sets of the form

The ℓ-spectrum of an Abelian ℓ-group with unit

- For an Abelian ℓ-group G with (order-)unit, we set $\mathrm{Spec}_\ell\, G \underset{\mathrm{def}}{=} \{P \mid P \text{ is a prime ideal of } G\}$, endowed with the topology whose closed sets are the sets of the form

$$\mathrm{Spec}_\ell(G, X) \underset{\mathrm{def}}{=} \{P \in \mathrm{Spec}_\ell\, G \mid X \subseteq P\}\,, \quad \text{for } X \subseteq G\,,$$

and we call it the ℓ-spectrum of G.

The ℓ-spectrum of an Abelian ℓ-group with unit

- For an Abelian ℓ-group G with (order-)unit, we set $\mathrm{Spec}_\ell\, G \underset{\mathrm{def}}{=} \{P \mid P \text{ is a prime ideal of } G\}$, endowed with the topology whose closed sets are the sets of the form

$$\mathrm{Spec}_\ell(G, X) \underset{\mathrm{def}}{=} \{P \in \mathrm{Spec}_\ell\, G \mid X \subseteq P\}\,, \quad \text{for } X \subseteq G\,,$$

  and we call it the ℓ-spectrum of G.
- It is well known that the ℓ-spectrum of any Abelian ℓ-group with unit is a spectral space.

The ℓ-spectrum of an Abelian ℓ-group with unit

- For an Abelian ℓ-group G with (order-)unit, we set $\mathrm{Spec}_\ell\, G \underset{\mathrm{def}}{=} \{P \mid P \text{ is a prime ideal of } G\}$, endowed with the topology whose closed sets are the sets of the form

$$\mathrm{Spec}_\ell(G,X) \underset{\mathrm{def}}{=} \{P \in \mathrm{Spec}_\ell\, G \mid X \subseteq P\}\,, \quad \text{for } X \subseteq G\,,$$

  and we call it the ℓ-spectrum of G.
- It is well known that the ℓ-spectrum of any Abelian ℓ-group with unit is a spectral space.
- It turns out that more is true!

Completely normal spectral spaces

- In any topological space X, the specialization preordering is defined by $x \leqslant y$ if $y \in \overline{\{x\}}$.

Completely normal spectral spaces

- In any topological space X, the specialization preordering is defined by $x \leqslant y$ if $y \in \overline{\{x\}}$.
- If X is spectral (or, much more generally, if X is T_0), then $\leqslant$ is an ordering (i.e., $x \leqslant y$ and $y \leqslant x$ implies that $x = y$).

Completely normal spectral spaces

- In any topological space X, the specialization preordering is defined by $x \leqslant y$ if $y \in \overline{\{x\}}$.
- If X is spectral (or, much more generally, if X is T_0), then $\leqslant$ is an ordering (i.e., $x \leqslant y$ and $y \leqslant x$ implies that $x = y$).
- A spectral space X is completely normal if $\leqslant$ is a root system, that is, $\{x, y\} \subseteq \overline{\{z\}} \Rightarrow (x \in \overline{\{y\}}$ or $y \in \overline{\{x\}})$.

Completely normal spectral spaces

- In any topological space X, the specialization preordering is defined by $x \leqslant y$ if $y \in \overline{\{x\}}$.
- If X is spectral (or, much more generally, if X is T_0), then $\leqslant$ is an ordering (i.e., $x \leqslant y$ and $y \leqslant x$ implies that $x = y$).
- A spectral space X is completely normal if $\leqslant$ is a root system, that is, $\{x, y\} \subseteq \overline{\{z\}} \Rightarrow (x \in \overline{\{y\}}$ or $y \in \overline{\{x\}})$.
- This is (properly) weaker than saying that every subspace of X is normal.

Completely normal spectral spaces

- In any topological space X, the specialization preordering is defined by $x \leqslant y$ if $y \in \overline{\{x\}}$.
- If X is spectral (or, much more generally, if X is T_0), then $\leqslant$ is an ordering (i.e., $x \leqslant y$ and $y \leqslant x$ implies that $x = y$).
- A spectral space X is completely normal if $\leqslant$ is a root system, that is, $\{x, y\} \subseteq \overline{\{z\}} \Rightarrow (x \in \overline{\{y\}}$ or $y \in \overline{\{x\}})$.
- This is (properly) weaker than saying that every subspace of X is normal.

Theorem (Monteiro 1954)

A spectral space X is completely normal iff its Stone dual $\overset{\circ}{\mathcal{K}}(X)$ is a completely normal lattice, that is,

Completely normal spectral spaces

- In any topological space X, the specialization preordering is defined by $x \leqslant y$ if $y \in \overline{\{x\}}$.
- If X is spectral (or, much more generally, if X is T_0), then $\leqslant$ is an ordering (i.e., $x \leqslant y$ and $y \leqslant x$ implies that $x = y$).
- A spectral space X is completely normal if $\leqslant$ is a root system, that is, $\{x, y\} \subseteq \overline{\{z\}} \Rightarrow (x \in \overline{\{y\}}$ or $y \in \overline{\{x\}})$.
- This is (properly) weaker than saying that every subspace of X is normal.

Theorem (Monteiro 1954)

A spectral space X is completely normal iff its Stone dual $\overset{\circ}{\mathcal{K}}(X)$ is a completely normal lattice, that is,

$$(\forall a, b)(\exists x, y)(a \vee b = a \vee y = x \vee b \text{ and } x \wedge y = 0)\,.$$

ℓ-spectra of Abelian ℓ-groups again

Theorem (Keimel 1971)

The ℓ-spectrum of any Abelian ℓ-group with unit is a completely normal spectral space.

ℓ-spectra of Abelian ℓ-groups again

Theorem (Keimel 1971)

The ℓ-spectrum of any Abelian ℓ-group with unit is a completely normal spectral space.

- The question, of characterizing ℓ-spectra, is open since then.

ℓ-spectra of Abelian ℓ-groups again

Theorem (Keimel 1971)

The ℓ-spectrum of any Abelian ℓ-group with unit is a completely normal spectral space.

- The question, of characterizing ℓ-spectra, is open since then.
- Equivalent to the MV-spectrum problem.

ℓ-spectra of Abelian ℓ-groups again

Theorem (Keimel 1971)

The ℓ-spectrum of any Abelian ℓ-group with unit is a completely normal spectral space.

- The question, of characterizing ℓ-spectra, is open since then.
- Equivalent to the MV-spectrum problem.

Theorem (Delzell and Madden 1994)

Not every completely normal spectral space is an ℓ-spectrum.

ℓ-spectra of Abelian ℓ-groups again

Theorem (Keimel 1971)

The ℓ-spectrum of any Abelian ℓ-group with unit is a completely normal spectral space.

- The question, of characterizing ℓ-spectra, is open since then.
- Equivalent to the MV-spectrum problem.

Theorem (Delzell and Madden 1994)

Not every completely normal spectral space is an ℓ-spectrum.

Delzell and Madden's example is not second countable (i.e., no countable basis of the topology): in fact, it has

card $\overset{\circ}{\mathcal{K}}(X) = \aleph_1$.

ℓ-spectra of countable Abelian ℓ-groups

Theorem (W. 2017)

Every second countable completely normal spectral space is homeomorphic to $\mathrm{Spec}_{\ell}\, G$ for some Abelian ℓ-group G with unit.

ℓ-spectra of countable Abelian ℓ-groups

Theorem (W. 2017)

Every second countable completely normal spectral space is homeomorphic to $\operatorname{Spec}_{\ell} G$ for some Abelian ℓ-group G with unit.

- Hence, Delzell and Madden's counterexample cannot be extended to the countable case.

ℓ-spectra of countable Abelian ℓ-groups

Theorem (W. 2017)

Every second countable completely normal spectral space is homeomorphic to $\mathrm{Spec}_\ell\, G$ for some Abelian ℓ-group G with unit.

- Hence, Delzell and Madden's counterexample cannot be extended to the countable case.
- Very rough outline of proof (of the countable case): start by observing that for any Abelian ℓ-group G with unit, the Stone dual of $\mathrm{Spec}_\ell\, G$ is $\mathrm{Id_c}\, G$, the lattice of all principal ℓ-ideals of G (ordered by $\subseteq$).

ℓ-spectra of countable Abelian ℓ-groups

Theorem (W. 2017)

Every second countable completely normal spectral space is homeomorphic to $\mathrm{Spec}_{\ell}\, G$ for some Abelian ℓ-group G with unit.

- Hence, Delzell and Madden's counterexample cannot be extended to the countable case.
- Very rough outline of proof (of the countable case): start by observing that for any Abelian ℓ-group G with unit, the Stone dual of $\mathrm{Spec}_{\ell}\, G$ is $\mathrm{Id}_{c}\, G$, the lattice of all principal ℓ-ideals of G (ordered by $\subseteq$).
- Since G has an order-unit, $\mathrm{Id}_{c}\, G$ is a bounded distributive lattice.

ℓ-spectra of countable Abelian ℓ-groups

Theorem (W. 2017)

Every second countable completely normal spectral space is homeomorphic to $\mathrm{Spec}_\ell\, G$ for some Abelian ℓ-group G with unit.

- Hence, Delzell and Madden's counterexample cannot be extended to the countable case.
- Very rough outline of proof (of the countable case): start by observing that for any Abelian ℓ-group G with unit, the Stone dual of $\mathrm{Spec}_\ell\, G$ is $\mathrm{Id_c}\, G$, the lattice of all principal ℓ-ideals of G (ordered by $\subseteq$).
- Since G has an order-unit, $\mathrm{Id_c}\, G$ is a bounded distributive lattice.
- Thus we must prove that every countable completely normal bounded distributive lattice D is $\cong \mathrm{Id_c}\, G$ for some Abelian ℓ-group G with unit.

Very rough outline of the proof of the countable case (cont'd)

- The idea is to construct a "nice" surjective $0,1$-lattice homomorphism $f\colon \operatorname{Id_c} F_\omega \twoheadrightarrow D$, where F_ω denotes the free Abelian ℓ-group on a countably infinite generating set.

Very rough outline of the proof of the countable case (cont'd)

- The idea is to construct a "nice" surjective $0, 1$-lattice homomorphism $f\colon \operatorname{Id_c} F_\omega \twoheadrightarrow D$, where F_ω denotes the free Abelian ℓ-group on a countably infinite generating set.
- "Nice" means that f should induce an isomorphism $\operatorname{Id_c}(F_\omega/I) \to D$, for the ℓ-ideal $I \underset{\text{def}}{=} \{x \in F_\omega \mid f(\langle x \rangle) = 0\}$.

Very rough outline of the proof of the countable case (cont'd)

- The idea is to construct a "nice" surjective $0,1$-lattice homomorphism $f\colon \mathrm{Id_c}\, F_\omega \twoheadrightarrow D$, where F_ω denotes the free Abelian ℓ-group on a countably infinite generating set.
- "Nice" means that f should induce an isomorphism $\mathrm{Id_c}(F_\omega/I) \to D$, for the ℓ-ideal $I \underset{\text{def}}{=} \{x \in F_\omega \mid f(\langle x \rangle) = 0\}$.
- It turns out that "nice" is easy to define!

Very rough outline of the proof of the countable case (cont'd)

- The idea is to construct a "nice" surjective $0,1$-lattice homomorphism $f\colon \mathrm{Id_c}\, F_\omega \twoheadrightarrow D$, where F_ω denotes the free Abelian ℓ-group on a countably infinite generating set.
- "Nice" means that f should induce an isomorphism $\mathrm{Id_c}(F_\omega/I) \to D$, for the ℓ-ideal $I \underset{\text{def}}{=} \{x \in F_\omega \mid f(\langle x\rangle) = 0\}$.
- It turns out that "nice" is easy to define!

Definition (closed maps)

For bounded distributive lattices A and B, a $0,1$-lattice homomorphism $f\colon A \to B$ is closed if whenever $a_0, a_1 \in A$ and $b \in B$, if $f(a_0) \leq f(a_1) \vee b$, then there exists $x \in A$ such that $a_0 \leq a_1 \vee x$ and $f(x) \leq b$.

Very rough outline of the proof of the countable case (cont'd)

- The idea is to construct a "nice" surjective $0,1$-lattice homomorphism $f \colon \mathrm{Id_c}\, F_\omega \twoheadrightarrow D$, where F_ω denotes the free Abelian ℓ-group on a countably infinite generating set.
- "Nice" means that f should induce an isomorphism $\mathrm{Id_c}(F_\omega/I) \to D$, for the ℓ-ideal $I \underset{\mathrm{def}}{=} \{x \in F_\omega \mid f(\langle x \rangle) = 0\}$.
- It turns out that "nice" is easy to define!

Definition (closed maps)

For bounded distributive lattices A and B, a $0,1$-lattice homomorphism $f \colon A \to B$ is closed if whenever $a_0, a_1 \in A$ and $b \in B$, if $f(a_0) \leq f(a_1) \vee b$, then there exists $x \in A$ such that $a_0 \leq a_1 \vee x$ and $f(x) \leq b$. Equivalently, the Stone dual map $\mathrm{Spec}\, f \colon \mathrm{Spec}\, B \to \mathrm{Spec}\, A$ is closed (i.e., it sends closed subsets to closed subsets).

Very rough outline of the proof of the countable case (further cont'd)

- The map $f\colon \operatorname{Id_c} F_\omega \to D$ is constructed as $f = \bigcup_{n<\omega} f_n$ (each $f_n \subseteq f_{n+1}$), where each $f_n\colon L_n \to D$ is a lattice homomorphism, for a carefully constructed finite sublattice L_n of $\operatorname{Id_c} F_\omega$.

Very rough outline of the proof of the countable case (further cont'd)

- The map $f\colon \operatorname{Id_c} F_\omega \to D$ is constructed as $f = \bigcup_{n<\omega} f_n$ (each $f_n \subseteq f_{n+1}$), where each $f_n\colon L_n \to D$ is a lattice homomorphism, for a carefully constructed finite sublattice L_n of $\operatorname{Id_c} F_\omega$.
- Due to a 2004 example of Di Nola and Grigolia, the L_n cannot all be completely normal.

Very rough outline of the proof of the countable case (further cont'd)

- The map $f\colon \operatorname{Id}_{\mathrm{c}} F_\omega \to D$ is constructed as $f = \bigcup_{n<\omega} f_n$ (each $f_n \subseteq f_{n+1}$), where each $f_n\colon L_n \to D$ is a lattice homomorphism, for a carefully constructed finite sublattice L_n of $\operatorname{Id}_{\mathrm{c}} F_\omega$.
- Due to a 2004 example of Di Nola and Grigolia, the L_n cannot all be completely normal.
- The finite distributive lattices L_n come out as special cases of the following construction.

Very rough outline of the proof of the countable case (further cont'd)

- The map $f\colon \operatorname{Id_c} F_\omega \to D$ is constructed as $f = \bigcup_{n<\omega} f_n$ (each $f_n \subseteq f_{n+1}$), where each $f_n\colon L_n \to D$ is a lattice homomorphism, for a carefully constructed finite sublattice L_n of $\operatorname{Id_c} F_\omega$.
- Due to a 2004 example of Di Nola and Grigolia, the L_n cannot all be completely normal.
- The finite distributive lattices L_n come out as special cases of the following construction.
- Let $\mathcal{H}$ be a set of closed hyperplanes of a topological vector space $\mathbb{E}$.

Very rough outline of the proof of the countable case (further cont'd)

- The map $f\colon \operatorname{Id}_c F_\omega \to D$ is constructed as $f = \bigcup_{n<\omega} f_n$ (each $f_n \subseteq f_{n+1}$), where each $f_n\colon L_n \to D$ is a lattice homomorphism, for a carefully constructed finite sublattice L_n of $\operatorname{Id}_c F_\omega$.
- Due to a 2004 example of Di Nola and Grigolia, the L_n cannot all be completely normal.
- The finite distributive lattices L_n come out as special cases of the following construction.
- Let $\mathcal{H}$ be a set of closed hyperplanes of a topological vector space $\mathbb{E}$.
- Each $H \in \mathcal{H}$ determines two open half-spaces H^+ and H^-.

Very rough outline of the proof of the countable case (further cont'd)

- The map $f\colon \mathrm{Id_c}\, F_\omega \to D$ is constructed as $f = \bigcup_{n<\omega} f_n$ (each $f_n \subseteq f_{n+1}$), where each $f_n\colon L_n \to D$ is a lattice homomorphism, for a carefully constructed finite sublattice L_n of $\mathrm{Id_c}\, F_\omega$.
- Due to a 2004 example of Di Nola and Grigolia, the L_n cannot all be completely normal.
- The finite distributive lattices L_n come out as special cases of the following construction.
- Let $\mathcal{H}$ be a set of closed hyperplanes of a topological vector space $\mathbb{E}$.
- Each $H \in \mathcal{H}$ determines two open half-spaces H^+ and H^-.
- Denote by $\mathrm{Op}(\mathcal{H})$ the 0, 1-sublattice of the powerset of $\mathbb{E}$ generated by $\{H^+ \mid H \in \mathcal{H}\} \cup \{H^- \mid H \in \mathcal{H}\}$.

Very rough outline of the proof of the countable case (further cont'd)

- The map $f \colon \mathrm{Id_c}\, F_\omega \to D$ is constructed as $f = \bigcup_{n<\omega} f_n$ (each $f_n \subseteq f_{n+1}$), where each $f_n \colon L_n \to D$ is a lattice homomorphism, for a carefully constructed finite sublattice L_n of $\mathrm{Id_c}\, F_\omega$.
- Due to a 2004 example of Di Nola and Grigolia, the L_n cannot all be completely normal.
- The finite distributive lattices L_n come out as special cases of the following construction.
- Let $\mathcal{H}$ be a set of closed hyperplanes of a topological vector space $\mathbb{E}$.
- Each $H \in \mathcal{H}$ determines two open half-spaces H^+ and H^-.
- Denote by $\mathrm{Op}(\mathcal{H})$ the $0,1$-sublattice of the powerset of $\mathbb{E}$ generated by $\{H^+ \mid H \in \mathcal{H}\} \cup \{H^- \mid H \in \mathcal{H}\}$.
- The subset $\mathrm{Op}^-(\mathcal{H}) \underset{\mathrm{def}}{=} \mathrm{Op}(\mathcal{H}) \setminus \{\mathbb{E}\}$ is a sublattice of $\mathrm{Op}(\mathcal{H})$.

Very rough outline of the proof of the countable case (coming to the end)

- The lattices L_n will have the form $\mathrm{Op}^{-}(\mathcal{H})$, for finite sets of integer hyperplanes in $\mathbb{E} \underset{\text{def}}{=} \mathbb{R}^{(\omega)}$.

Very rough outline of the proof of the countable case (coming to the end)

- The lattices L_n will have the form $\mathrm{Op}^-(\mathcal{H})$, for finite sets of integer hyperplanes in $\mathbb{E} \underset{\text{def}}{=} \mathbb{R}^{(\omega)}$.
- This is made possible by the Baker-Beynon duality, which implies that $\mathrm{Id_c}\, F_\omega \cong \mathrm{Op}^-(\mathcal{H}_\mathbb{Z})$, where $\mathcal{H}_\mathbb{Z}$ denotes the (countable) set of all integer hyperplanes of $\mathbb{R}^{(\omega)}$.

Very rough outline of the proof of the countable case (coming to the end)

- The lattices L_n will have the form $\mathrm{Op}^-(\mathcal{H})$, for finite sets of integer hyperplanes in $\mathbb{E} \underset{\text{def}}{=} \mathbb{R}^{(\omega)}$.
- This is made possible by the Baker-Beynon duality, which implies that $\mathrm{Id_c}\, F_\omega \cong \mathrm{Op}^-(\mathcal{H}_\mathbb{Z})$, where $\mathcal{H}_\mathbb{Z}$ denotes the (countable) set of all integer hyperplanes of $\mathbb{R}^{(\omega)}$.
- Each enlargement step, from f_n to f_{n+1}, corrects one of the following three types of defects:

Very rough outline of the proof of the countable case (coming to the end)

- The lattices L_n will have the form $\mathrm{Op}^-(\mathcal{H})$, for finite sets of integer hyperplanes in $\mathbb{E} \underset{\text{def}}{=} \mathbb{R}^{(\omega)}$.
- This is made possible by the Baker-Beynon duality, which implies that $\mathrm{Id_c}\, F_\omega \cong \mathrm{Op}^-(\mathcal{H}_\mathbb{Z})$, where $\mathcal{H}_\mathbb{Z}$ denotes the (countable) set of all integer hyperplanes of $\mathbb{R}^{(\omega)}$.
- Each enlargement step, from f_n to f_{n+1}, corrects one of the following three types of defects:
  - (hard) f_n is not defined everywhere: then add a pair (H^+, H^-) to the domain of f_n;

Very rough outline of the proof of the countable case (coming to the end)

- The lattices L_n will have the form $\mathrm{Op}^-(\mathcal{H})$, for finite sets of integer hyperplanes in $\mathbb{E} \underset{\mathrm{def}}{=} \mathbb{R}^{(\omega)}$.
- This is made possible by the Baker-Beynon duality, which implies that $\mathrm{Id_c}\, F_\omega \cong \mathrm{Op}^-(\mathcal{H}_\mathbb{Z})$, where $\mathcal{H}_\mathbb{Z}$ denotes the (countable) set of all integer hyperplanes of $\mathbb{R}^{(\omega)}$.
- Each enlargement step, from f_n to f_{n+1}, corrects one of the following three types of defects:
  - (hard) f_n is not defined everywhere: then add a pair (H^+, H^-) to the domain of f_n;
  - (easy, but infinite dimension needed!) f_n is not surjective: then add an element to the range of f_n;

Very rough outline of the proof of the countable case (coming to the end)

- The lattices L_n will have the form $\mathrm{Op}^-(\mathcal{H})$, for finite sets of integer hyperplanes in $\mathbb{E} \underset{\text{def}}{=} \mathbb{R}^{(\omega)}$.
- This is made possible by the Baker-Beynon duality, which implies that $\mathrm{Id_c}\, F_\omega \cong \mathrm{Op}^-(\mathcal{H}_\mathbb{Z})$, where $\mathcal{H}_\mathbb{Z}$ denotes the (countable) set of all integer hyperplanes of $\mathbb{R}^{(\omega)}$.
- Each enlargement step, from f_n to f_{n+1}, corrects one of the following three types of defects:
  - (hard) f_n is not defined everywhere: then add a pair (H^+, H^-) to the domain of f_n;
  - (easy, but infinite dimension needed!) f_n is not surjective: then add an element to the range of f_n;
  - (hardest) f_n is not closed: then let f_{n+1} correct a closure defect $f_n(A_0) \leq f_n(A_1) \vee \gamma$.

Very rough outline of the proof of the countable case (coming to the end)

- The lattices L_n will have the form $\mathrm{Op}^-(\mathcal{H})$, for finite sets of integer hyperplanes in $\mathbb{E} \underset{\text{def}}{=} \mathbb{R}^{(\omega)}$.
- This is made possible by the Baker-Beynon duality, which implies that $\mathrm{Id_c}\, F_\omega \cong \mathrm{Op}^-(\mathcal{H}_\mathbb{Z})$, where $\mathcal{H}_\mathbb{Z}$ denotes the (countable) set of all integer hyperplanes of $\mathbb{R}^{(\omega)}$.
- Each enlargement step, from f_n to f_{n+1}, corrects one of the following three types of defects:
  - (hard) f_n is not defined everywhere: then add a pair (H^+, H^-) to the domain of f_n;
  - (easy, but infinite dimension needed!) f_n is not surjective: then add an element to the range of f_n;
  - (hardest) f_n is not closed: then let f_{n+1} correct a closure defect $f_n(A_0) \leq f_n(A_1) \vee \gamma$.
- A crucial observation is that each $\mathrm{Op}(\mathcal{H})$ is a Heyting subalgebra of the Heyting algebra of all open subsets of $\mathbb{E}$.

Loose ends on ℓ-spectra

- Say that a lattice D is ℓ-representable if it is $\cong \mathrm{Id_c}\, G$ for some Abelian ℓ-group G with unit.

Loose ends on ℓ-spectra

- Say that a lattice D is ℓ-representable if it is $\cong \mathrm{Id_c}\, G$ for some Abelian ℓ-group G with unit.
- Equivalently, D is the Stone dual of $\mathrm{Spec}_\ell\, G$ for some Abelian ℓ-group G with unit.

Loose ends on ℓ-spectra

- Say that a lattice D is ℓ-representable if it is $\cong \mathrm{Id_c}\, G$ for some Abelian ℓ-group G with unit.
- Equivalently, D is the Stone dual of $\mathrm{Spec}_\ell\, G$ for some Abelian ℓ-group G with unit.
- By the above, a countable bounded distributive lattice is ℓ-representable iff it is completely normal.

Loose ends on ℓ-spectra

- Say that a lattice D is ℓ-representable if it is $\cong \mathrm{Id_c}\, G$ for some Abelian ℓ-group G with unit.
- Equivalently, D is the Stone dual of $\mathrm{Spec}_\ell\, G$ for some Abelian ℓ-group G with unit.
- By the above, a countable bounded distributive lattice is ℓ-representable iff it is completely normal.
- By Delzell and Madden's example, this fails for uncountable lattices. In fact,

Loose ends on ℓ-spectra

- Say that a lattice D is ℓ-representable if it is $\cong \mathrm{Id_c}\, G$ for some Abelian ℓ-group G with unit.
- Equivalently, D is the Stone dual of $\mathrm{Spec}_\ell\, G$ for some Abelian ℓ-group G with unit.
- By the above, a countable bounded distributive lattice is ℓ-representable iff it is completely normal.
- By Delzell and Madden's example, this fails for uncountable lattices. In fact,

Theorem (W. 2017)

The class of all ℓ-representable lattices is not $\mathscr{L}_{\infty,\omega}$-definable

Loose ends on ℓ-spectra

- Say that a lattice D is ℓ-representable if it is $\cong \mathrm{Id_c}\, G$ for some Abelian ℓ-group G with unit.
- Equivalently, D is the Stone dual of $\mathrm{Spec}_{\ell}\, G$ for some Abelian ℓ-group G with unit.
- By the above, a countable bounded distributive lattice is ℓ-representable iff it is completely normal.
- By Delzell and Madden's example, this fails for uncountable lattices. In fact,

Theorem (W. 2017)

The class of all ℓ-representable lattices is not $\mathscr{L}_{\infty,\omega}$-definable (thus, *a fortiori*, not first-order definable).

Loose ends on ℓ-spectra

- Say that a lattice D is ℓ-representable if it is $\cong \mathrm{Id_c}\, G$ for some Abelian ℓ-group G with unit.
- Equivalently, D is the Stone dual of $\mathrm{Spec}_\ell\, G$ for some Abelian ℓ-group G with unit.
- By the above, a countable bounded distributive lattice is ℓ-representable iff it is completely normal.
- By Delzell and Madden's example, this fails for uncountable lattices. In fact,

Theorem (W. 2017)

The class of all ℓ-representable lattices is not $\mathscr{L}_{\infty,\omega}$-definable (thus, *a fortiori*, not first-order definable).

Analogous result for $\mathscr{L}_{\infty,\lambda}$ (for any infinite cardinal λ): proof currently under verification.

Cones, prime cones, real spectrum

- The real spectrum was introduced in 1981, by Coste and Coste-Roy, as an ordered analogue of the Zariski spectrum of a commutative unital ring.

Cones, prime cones, real spectrum

- The real spectrum was introduced in 1981, by Coste and Coste-Roy, as an ordered analogue of the Zariski spectrum of a commutative unital ring.
- Let A be a commutative unital ring (not necessarily ordered). A cone of A is a subset C of A such that $C + C \subseteq C$, $C \cdot C \subseteq C$, and $a^2 \in C$ whenever $a \in A$.

Cones, prime cones, real spectrum

- The real spectrum was introduced in 1981, by Coste and Coste-Roy, as an ordered analogue of the Zariski spectrum of a commutative unital ring.
- Let A be a commutative unital ring (not necessarily ordered). A cone of A is a subset C of A such that $C + C \subseteq C$, $C \cdot C \subseteq C$, and $a^2 \in C$ whenever $a \in A$.
- A cone C is prime if $C \cap (-C)$ is a prime ideal of A and $A = C \cup (-C)$.

Cones, prime cones, real spectrum

- The real spectrum was introduced in 1981, by Coste and Coste-Roy, as an ordered analogue of the Zariski spectrum of a commutative unital ring.
- Let A be a commutative unital ring (not necessarily ordered). A cone of A is a subset C of A such that $C + C \subseteq C$, $C \cdot C \subseteq C$, and $a^2 \in C$ whenever $a \in A$.
- A cone C is prime if $C \cap (-C)$ is a prime ideal of A and $A = C \cup (-C)$.
- We endow the set $\mathrm{Spec_r}\, A$ of all prime cones of A with the topology generated by the sets $\{P \in \mathrm{Spec_r}\, A \mid a \notin P\}$, for $a \in A$.

Cones, prime cones, real spectrum

- The real spectrum was introduced in 1981, by Coste and Coste-Roy, as an ordered analogue of the Zariski spectrum of a commutative unital ring.
- Let A be a commutative unital ring (not necessarily ordered). A cone of A is a subset C of A such that $C + C \subseteq C$, $C \cdot C \subseteq C$, and $a^2 \in C$ whenever $a \in A$.
- A cone C is prime if $C \cap (-C)$ is a prime ideal of A and $A = C \cup (-C)$.
- We endow the set $\operatorname{Spec}_{\mathrm{r}} A$ of all prime cones of A with the topology generated by the sets $\{P \in \operatorname{Spec}_{\mathrm{r}} A \mid a \notin P\}$, for $a \in A$. The topological space thus obtained is called the real spectrum of A.

Cones, prime cones, real spectrum

- The real spectrum was introduced in 1981, by Coste and Coste-Roy, as an ordered analogue of the Zariski spectrum of a commutative unital ring.
- Let A be a commutative unital ring (not necessarily ordered). A cone of A is a subset C of A such that $C + C \subseteq C$, $C \cdot C \subseteq C$, and $a^2 \in C$ whenever $a \in A$.
- A cone C is prime if $C \cap (-C)$ is a prime ideal of A and $A = C \cup (-C)$.
- We endow the set $\mathrm{Spec_r}\, A$ of all prime cones of A with the topology generated by the sets $\{P \in \mathrm{Spec_r}\, A \mid a \notin P\}$, for $a \in A$. The topological space thus obtained is called the real spectrum of A.
- It turns out that $\mathrm{Spec_r}\, A$ is a completely normal spectral space, for any commutative unital ring A.

Characterizing problem of real spectra

Problem (Keimel 1991)

Characterize real spectra of commutative unital rings.

Characterizing problem of real spectra

Problem (Keimel 1991)

Characterize real spectra of commutative unital rings.

- The countable case of the problem above (i.e., for second countable spaces) is still open.

Characterizing problem of real spectra

Problem (Keimel 1991)

Characterize real spectra of commutative unital rings.

- The countable case of the problem above (i.e., for second countable spaces) is still open.
- Negative answer in the uncountable case:

Characterizing problem of real spectra

Problem (Keimel 1991)

Characterize real spectra of commutative unital rings.

- The countable case of the problem above (i.e., for second countable spaces) is still open.
- Negative answer in the uncountable case:

Theorem (Delzell and Madden 1994)

Not every completely normal spectral space is a real spectrum.

Characterizing problem of real spectra

Problem (Keimel 1991)

Characterize real spectra of commutative unital rings.

- The countable case of the problem above (i.e., for second countable spaces) is still open.
- Negative answer in the uncountable case:

Theorem (Delzell and Madden 1994)

Not every completely normal spectral space is a real spectrum.

Theorem (Mellor and Tressl 2012)

For any infinite cardinal λ, there is no $\mathscr{L}_{\infty,\lambda}$-characterization of the Stone duals of real spectra of commutative unital rings.

Subspaces of ℓ-spectra and real spectra

It is known that every closed subspace of an ℓ-spectrum (resp., real spectrum) is an ℓ-spectrum (resp., real spectrum).

Subspaces of ℓ-spectra and real spectra

It is known that every closed subspace of an ℓ-spectrum (resp., real spectrum) is an ℓ-spectrum (resp., real spectrum).

Theorem (W. 2017)

Not every spectral subspace of an ℓ-spectrum (resp., real spectrum) is an ℓ-spectrum (resp., real spectrum).

Subspaces of ℓ-spectra and real spectra

It is known that every closed subspace of an ℓ-spectrum (resp., real spectrum) is an ℓ-spectrum (resp., real spectrum).

Theorem (W. 2017)

Not every spectral subspace of an ℓ-spectrum (resp., real spectrum) is an ℓ-spectrum (resp., real spectrum).

Problem (W. 2017)

Is a retract of an ℓ-spectrum also an ℓ-spectrum? Same question for real spectra.

Comparing spectra

- For any class $\mathbf{X}$ of spectral spaces, denote by $\mathbf{SX}$ the class of all spectral subspaces of members of $\mathbf{X}$.

Comparing spectra

- For any class $\mathbf{X}$ of spectral spaces, denote by $\mathbf{SX}$ the class of all spectral subspaces of members of $\mathbf{X}$.
- Then introduce the following classes of spectral spaces:

Comparing spectra

- For any class **X** of spectral spaces, denote by **SX** the class of all spectral subspaces of members of **X**.
- Then introduce the following classes of spectral spaces:
  - **CN**, the class of all completely normal spectral spaces;

Comparing spectra

- For any class **X** of spectral spaces, denote by **SX** the class of all spectral subspaces of members of **X**.
- Then introduce the following classes of spectral spaces:
  - **CN**, the class of all completely normal spectral spaces;
  - $\boldsymbol{\ell}$, the class of all ℓ-spectra of Abelian ℓ-groups with unit;

Comparing spectra

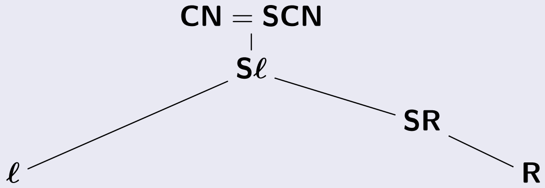
- For any class **X** of spectral spaces, denote by **SX** the class of all spectral subspaces of members of **X**.
- Then introduce the following classes of spectral spaces:
  - **CN**, the class of all completely normal spectral spaces;
  - $\boldsymbol{\ell}$, the class of all ℓ-spectra of Abelian ℓ-groups with unit;
  - **R**, the class of all real spectra of commutative unital rings.

Comparing spectra

- For any class $\mathbf{X}$ of spectral spaces, denote by $\mathbf{SX}$ the class of all spectral subspaces of members of $\mathbf{X}$.
- Then introduce the following classes of spectral spaces:
  - $\mathbf{CN}$, the class of all completely normal spectral spaces;
  - $\boldsymbol{\ell}$, the class of all ℓ-spectra of Abelian ℓ-groups with unit;
  - $\mathbf{R}$, the class of all real spectra of commutative unital rings.

Theorem (W. 2017)

All containments and non-containments of the following picture are valid:



- All the separating counterexamples, intervening in the result above, have size $\aleph_1$, except for the counterexample witnessing $\mathbf{S}\boldsymbol{\ell} \subsetneq \mathbf{CN}$, which has size $\aleph_2$.

- All the separating counterexamples, intervening in the result above, have size $\aleph_1$, except for the counterexample witnessing $\mathbf{S}\ell \subsetneqq \mathbf{CN}$, which has size $\aleph_2$.
- Most of the examples constructed for the theorem above involve the construction of condensate (Gillibert and W. 2011), which turns diagram counterexamples to object counterexamples, with a jump of alephs corresponding to the order-dimension of the poset indexing the diagram (thus $\aleph_1$, $\aleph_2$, and so on).

A counterexample witnessing $\mathbf{R} \not\subseteq \ell$

- Knebusch and Scheiderer proved in 1989 that for any homomorphism $f\colon R \to S$ of commutative unital rings, the map $\operatorname{Spec}_{\mathrm{r}} f\colon \operatorname{Spec}_{\mathrm{r}} S \to \operatorname{Spec}_{\mathrm{r}} R$ is convex, that is, whenever $Q_0 \subseteq Q_1$ in $\operatorname{Spec}_{\mathrm{r}} S$, $P \in \operatorname{Spec}_{\mathrm{r}} R$, and $f^{-1}Q_0 \subseteq P \subseteq f^{-1}Q_1$, there exists $Q \in \operatorname{Spec}_{\mathrm{r}} S$ such that $Q_0 \subseteq Q \subseteq Q_1$ and $P = f^{-1}Q$.

A counterexample witnessing $\mathbf{R} \not\subseteq \ell$

- Knebusch and Scheiderer proved in 1989 that for any homomorphism $f \colon R \to S$ of commutative unital rings, the map $\operatorname{Spec}_{\mathrm{r}} f \colon \operatorname{Spec}_{\mathrm{r}} S \to \operatorname{Spec}_{\mathrm{r}} R$ is convex, that is, whenever $Q_0 \subseteq Q_1$ in $\operatorname{Spec}_{\mathrm{r}} S$, $P \in \operatorname{Spec}_{\mathrm{r}} R$, and $f^{-1}Q_0 \subseteq P \subseteq f^{-1}Q_1$, there exists $Q \in \operatorname{Spec}_{\mathrm{r}} S$ such that $Q_0 \subseteq Q \subseteq Q_1$ and $P = f^{-1}Q$.
- Let K be any countable, non-Archimedean real-closed field, and set

$$A \underset{\mathrm{def}}{=} \{x \in K \mid (\exists n < \omega)(-n \cdot 1 \leq x \leq n \cdot 1\}\,.$$

A counterexample witnessing $\mathbf{R} \not\subseteq \ell$

- Knebusch and Scheiderer proved in 1989 that for any homomorphism $f\colon R \to S$ of commutative unital rings, the map $\operatorname{Spec_r} f\colon \operatorname{Spec_r} S \to \operatorname{Spec_r} R$ is convex, that is, whenever $Q_0 \subseteq Q_1$ in $\operatorname{Spec_r} S$, $P \in \operatorname{Spec_r} R$, and $f^{-1}Q_0 \subseteq P \subseteq f^{-1}Q_1$, there exists $Q \in \operatorname{Spec_r} S$ such that $Q_0 \subseteq Q \subseteq Q_1$ and $P = f^{-1}Q$.
- Let K be any countable, non-Archimedean real-closed field, and set

$$A \underset{\mathrm{def}}{=} \{x \in K \mid (\exists n < \omega)(-n \cdot 1 \leq x \leq n \cdot 1\}\,.$$

- The counterexample is the ring R of all almost constant families $(x_\xi \mid \xi < \omega_1) \in K^{\omega_1}$ such that $x_\infty \in A$: there is no Abelian ℓ-group G such that $\operatorname{Spec_r} R \cong \operatorname{Spec}_\ell G$. This is partly due to Knebusch and Scheiderer's result.

A counterexample witnessing **SR** $\not\subseteq$ **R**

- Start with a countable real-closed domain with exactly three prime ideals $\{0\} \subsetneqq P_1 \subsetneqq P_2$. Then consider the ring E of all almost constant ω_1-indexed families of elements of A.

A counterexample witnessing $\mathbf{SR} \not\subseteq \mathbf{R}$

- Start with a countable real-closed domain with exactly three prime ideals $\{0\} \subsetneqq P_1 \subsetneqq P_2$. Then consider the ring E of all almost constant ω_1-indexed families of elements of A.
- Define $\varphi\colon \mathbf{4} \underset{\text{def}}{=} \{0,1,2,3\} \twoheadrightarrow \mathbf{3} \underset{\text{def}}{=} \{0,1,2\}$ as the Stone dual of the (non-convex) map $\{1,2\} \to \{1,2,3\}$, $1 \mapsto 1$, $2 \mapsto 3$. Hence $\varphi(0)=0$, $\varphi(1)=\varphi(2)=1$, $\varphi(3)=2$.

A counterexample witnessing $\mathbf{SR} \not\subseteq \mathbf{R}$

- Start with a countable real-closed domain with exactly three prime ideals $\{0\} \subsetneqq P_1 \subsetneqq P_2$. Then consider the ring E of all almost constant ω_1-indexed families of elements of A.
- Define $\varphi \colon \mathbf{4} \underset{\text{def}}{=} \{0, 1, 2, 3\} \twoheadrightarrow \mathbf{3} \underset{\text{def}}{=} \{0, 1, 2\}$ as the Stone dual of the (non-convex) map $\{1, 2\} \to \{1, 2, 3\}$, $1 \mapsto 1$, $2 \mapsto 3$. Hence $\varphi(0) = 0$, $\varphi(1) = \varphi(2) = 1$, $\varphi(3) = 2$.
- The lattice $\operatorname{Cond}(\varphi, \omega_1) \underset{\text{def}}{=} \{(x, y) \in \mathbf{4} \times \mathbf{3}^{\omega_1} \mid y_\xi = \varphi(x) \text{ for all but finitely many } \xi\}$ is not the dual space of any real spectrum (because of Knebusch and Scheiderer's result).

A counterexample witnessing $\mathbf{SR} \not\subseteq \mathbf{R}$

- Start with a countable real-closed domain with exactly three prime ideals $\{0\} \subsetneqq P_1 \subsetneqq P_2$. Then consider the ring E of all almost constant ω_1-indexed families of elements of A.
- Define $\varphi \colon \mathbf{4} \underset{\text{def}}{=} \{0,1,2,3\} \twoheadrightarrow \mathbf{3} \underset{\text{def}}{=} \{0,1,2\}$ as the Stone dual of the (non-convex) map $\{1,2\} \to \{1,2,3\}$, $1 \mapsto 1$, $2 \mapsto 3$. Hence $\varphi(0) = 0$, $\varphi(1) = \varphi(2) = 1$, $\varphi(3) = 2$.
- The lattice $\operatorname{Cond}(\varphi, \omega_1) \underset{\text{def}}{=} \{(x,y) \in \mathbf{4} \times \mathbf{3}^{\omega_1} \mid y_\xi = \varphi(x)$ for all but finitely many $\xi\}$ is not the dual space of any real spectrum (because of Knebusch and Scheiderer's result).
- However, $\operatorname{Cond}(\varphi, \omega_1)$ is a homomorphic image of the dual space of the real spectrum of E.

A counterexample witnessing $\ell \not\subseteq \mathbf{SR}$

- For any chain Λ, denote by $\mathbb{Z}\langle\Lambda\rangle$ the lexicographical power of $\mathbb{Z}$ by Λ: hence $\alpha < \beta$ in Λ implies that $n\alpha < \beta$ in $\mathbb{Z}\langle\Lambda\rangle$ for every integer n.

A counterexample witnessing $\boldsymbol{\ell} \not\subseteq \mathbf{SR}$

- For any chain Λ, denote by $\mathbb{Z}\langle\Lambda\rangle$ the lexicographical power of $\mathbb{Z}$ by Λ: hence $\alpha < \beta$ in Λ implies that $n\alpha < \beta$ in $\mathbb{Z}\langle\Lambda\rangle$ for every integer n.
- Denote by F the Abelian ℓ-group defined by generators a and b subjected to the relations $a \geq 0$ and $b \geq 0$.

A counterexample witnessing $\ell \not\subseteq$ SR

- For any chain Λ, denote by $\mathbb{Z}\langle\Lambda\rangle$ the lexicographical power of $\mathbb{Z}$ by Λ: hence $\alpha < \beta$ in Λ implies that $n\alpha < \beta$ in $\mathbb{Z}\langle\Lambda\rangle$ for every integer n.
- Denote by F the Abelian ℓ-group defined by generators a and b subjected to the relations $a \geq 0$ and $b \geq 0$.
- The counterexample is the lexicographical product $G \underset{\text{def}}{=} \mathbb{Z}\langle\omega_1^{\text{op}}\rangle \times_{\text{lex}} F$:

A counterexample witnessing $\boldsymbol{\ell} \not\subseteq$ **SR**

- For any chain Λ, denote by $\mathbb{Z}\langle\Lambda\rangle$ the lexicographical power of $\mathbb{Z}$ by Λ: hence $\alpha < \beta$ in Λ implies that $n\alpha < \beta$ in $\mathbb{Z}\langle\Lambda\rangle$ for every integer n.
- Denote by F the Abelian ℓ-group defined by generators a and b subjected to the relations $a \geq 0$ and $b \geq 0$.
- The counterexample is the lexicographical product $G \underset{\text{def}}{=} \mathbb{Z}\langle\omega_1^{\mathrm{op}}\rangle \times_{\mathrm{lex}} F$:
- $\mathrm{Spec}_\ell\, G$ cannot be embedded, as a spectral subspace, into the real spectrum of any commutative unital ring.

A counterexample witnessing $\mathbf{CN} \not\subseteq \mathbf{S}\ell$

- Start observing that any homomorphic image of the Stone dual of any $\mathrm{Spec}_\ell\, G$ satisfies the following family of infinitary statements:

A counterexample witnessing $\mathbf{CN} \not\subseteq \mathbf{S}\ell$

- Start observing that any homomorphic image of the Stone dual of any $\mathrm{Spec}_\ell\, G$ satisfies the following family of infinitary statements:
- For any family $(a_i \mid i \in I)$, there are elements $c_{i,j}$ such that each $a_i = (a_i \wedge a_j) \vee c_{i,j}$, each $c_{i,j} \wedge c_{j,i} = 0$, and each $c_{i,k} \leq c_{i,j} \vee c_{j,k}$.

A counterexample witnessing $\mathbf{CN} \not\subseteq \mathbf{S}\ell$

- Start observing that any homomorphic image of the Stone dual of any $\mathrm{Spec}_\ell\, G$ satisfies the following family of infinitary statements:
- For any family $(a_i \mid i \in I)$, there are elements $c_{i,j}$ such that each $a_i = (a_i \wedge a_j) \vee c_{i,j}$, each $c_{i,j} \wedge c_{j,i} = 0$, and each $c_{i,k} \leq c_{i,j} \vee c_{j,k}$.
- Consider the variety $\mathcal{V}$, in the similarity type $(0, 1, \vee, \wedge, \smallsetminus)$, whose identities are those of bounded distributive lattices, together with the additional identities

$$x = (x \wedge y) \vee (x \smallsetminus y)\,; \quad (x \smallsetminus y) \wedge (y \smallsetminus x) = 0\,.$$

A counterexample witnessing $\mathbf{CN} \not\subseteq \mathbf{S}\ell$

- Start observing that any homomorphic image of the Stone dual of any $\mathrm{Spec}_\ell\, G$ satisfies the following family of infinitary statements:
- For any family $(a_i \mid i \in I)$, there are elements $c_{i,j}$ such that each $a_i = (a_i \wedge a_j) \vee c_{i,j}$, each $c_{i,j} \wedge c_{j,i} = 0$, and each $c_{i,k} \leq c_{i,j} \vee c_{j,k}$.
- Consider the variety $\mathcal{V}$, in the similarity type $(0, 1, \vee, \wedge, \smallsetminus)$, whose identities are those of bounded distributive lattices, together with the additional identities

$$x = (x \wedge y) \vee (x \smallsetminus y)\,; \quad (x \smallsetminus y) \wedge (y \smallsetminus x) = 0\,.$$

- The counterexample is (the Stone dual of) $\mathrm{Fr}_{\mathcal{V}}(\omega_2)$.

A counterexample witnessing $\mathbf{CN} \not\subseteq \mathbf{S}\ell$

- Start observing that any homomorphic image of the Stone dual of any $\mathrm{Spec}_\ell\, G$ satisfies the following family of infinitary statements:
- For any family $(a_i \mid i \in I)$, there are elements $c_{i,j}$ such that each $a_i = (a_i \wedge a_j) \vee c_{i,j}$, each $c_{i,j} \wedge c_{j,i} = 0$, and each $c_{i,k} \leq c_{i,j} \vee c_{j,k}$.
- Consider the variety $\mathcal{V}$, in the similarity type $(0, 1, \vee, \wedge, \smallsetminus)$, whose identities are those of bounded distributive lattices, together with the additional identities

$$x = (x \wedge y) \vee (x \smallsetminus y)\,; \quad (x \smallsetminus y) \wedge (y \smallsetminus x) = 0\,.$$

- The counterexample is (the Stone dual of) $\mathrm{Fr}_{\mathcal{V}}(\omega_2)$.
- It works because of Kuratowski's Free Set Theorem.

Thanks for your attention!