Anti-elementarity for ranges of functors

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Main references

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2. F. Wehrung, *From non-commutative diagrams to anti-elementary classes*, hal-02000602, preprint, 2019

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- Tractability usually understood in logical sense: typically, describability *via* a class of (possibly infinitary) first-order sentences.
- We show how to prove that many "natural" functor ranges are intractable in the above sense.

Examples illustrating tractability

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- Refinement monoids are tractable within commutative monoids (again via a single first-order sentence):

$$(\forall a_0, a_1, b_0, b_1)\big(a_0 + a_1 = b_0 + b_1 \Rightarrow (\exists c_{00}, c_{01}, c_{10}, c_{11}) \\ (a_0 = c_{00} + c_{01} \;\&\; a_1 = c_{10} + c_{11} \;\&\; \\ b_0 = c_{00} + c_{10} \;\&\; b_1 = c_{01} + c_{11})\big)\,.$$

Infinitary languages

- Let Σ be a first-order language: collection of symbols of relations and functions, each given with an arity (possibly 0 for functions). *Example*: $(\underset{(0)}{0}, \underset{(0)}{1}, \underset{(2)}{+}, \underset{(2)}{\cdot}, \underset{(2)}{\leq})$ ("language of partially ordered unital rings").

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- First-order formulas are obtained by closing atomic formulas under finite conjunctions / disjunctions, negations, and $\exists$ / $\forall$ quantifiers.
- For infinite cardinals with $\kappa \geq \lambda$, $\mathscr{L}_{\kappa\lambda}$ is defined similarly, with conjunctions / disjunctions of less than κ formulas and quantifiers over strings of less than λ variables.

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- Finiteness can be expressed by a single $\mathscr{L}_{\omega_1\omega}$ sentence:

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- Archimedean property (for partially ordered Abelian groups) can be expressed by an $\mathscr{L}_{\omega_1\omega}$ sentence:

$$(\forall x,y)\Big(\bigwedge\!\!\!\bigwedge_{n<\omega}(nx\leq y)\Rightarrow x\leq 0\Big)\,.$$

A categorical statement implying elementarity

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Proposition (W 2019)

Let λ be an infinite regular cardinal, let Σ be a first-order language, let Ω be a set, and let $\Gamma\colon \mathfrak{P}_{\text{inj}}(\Omega) \to \mathbf{Str}\,\Sigma$ be a λ-continuous functor. Then for every $f\colon X \rightarrowtail Y$ in $\mathfrak{P}_{\text{inj}}(\Omega)$ with $\operatorname{card} X \geq \lambda$, $\Gamma(f)$ is an $\mathscr{L}_{\infty\lambda}$-elementary embedding from $\Gamma(X)$ into $\Gamma(Y)$.

Anti-elementarity

Definition

A class $\mathcal{C}$ of objects, in a category $\mathcal{S}$, is anti-elementary if there are arbitrarily large cardinals $\lambda < \kappa$ with λ-continuous functors $\Gamma \colon \mathfrak{P}_{\mathrm{inj}}(\kappa) \to \mathcal{S}$ such that $\Gamma(\lambda) \in \mathcal{C}$ and $\Gamma(\kappa) \notin \mathcal{C}$.

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- We shall outline a method making it possible to establish anti-elementarity for many classes. Those classes will always be ranges of functors.

A few useful categories

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- $\mathbf{CMon} \stackrel{\text{def}}{=}$ category of all commutative monoids with monoid homomorphisms.

Functors for which the method works

Theorem (W 2019)

The ranges of the following functors are all anti-elementary:

1. $\mathrm{Cs_c} \colon \mathcal{G} \to \mathbf{DLat}_0$, $G \mapsto$ lattice of all order-convex ℓ-subgroups of the ℓ-group G; for any class $\mathcal{G}$ of ℓ-groups containing all Archimedean ones.
2. $\mathrm{Id_c} \colon \mathcal{R} \to \mathbf{SLat}_0$, $R \mapsto$ semilattice of all finitely generated two-sided ideals of R, for many classes $\mathcal{R}$ of unital rings, including all unital regular rings and all unital rings.
3. $\mathrm{V} \colon \mathcal{R} \to \mathbf{CMon}$, $R \mapsto$ nonstable K_0-theory $\mathrm{V}(R)$ of R, for many classes $\mathcal{R}$ of unital rings, including all unital regular rings and all C*-algebras of real rank zero.

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  1. $\Phi\vec{A}^I$ (now a P^I-indexed diagram) is a commutative diagram for every set I (we say that $\vec{A}$ is Φ-commutative);
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Under quite general conditions, the above implies that the range of Φ is anti-elementary.

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$$P\langle U\rangle \overset{\mathrm{def}}{=} \left\{ (a,x) \mid a \in P\,,\ x\colon X \to U\,,\ X \text{ finite}\,,\ a = \bigvee X \right\}$$

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  with $(a,x) \leq (b,y)$ iff $a \leq b$ and y extends x, and additional map $\partial\colon P\langle U\rangle \to P$, $(a,x) \mapsto a$.
- Owing to an additional property of $P\langle U\rangle$ (we say that it is a "pseudo join-semilattice"), we say that $(P\langle U\rangle, \partial)$ is said to be a norm-covering of P

P-scaled Boolean algebras

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- Formally, a structure $\boldsymbol{B} = (B, (B^{(p)} \mid p \in P))$ is a P-scaled Boolean algebra if B is a Boolean algebra, each $B^{(p)}$ is an ideal of B, $1 \in \bigvee_{p \in P} B^{(p)}$, and for all $p, q \in P$, $B^{(p)} \cap B^{(q)} = \bigvee_{r \geq p,q} B^{(r)}$

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- The category $\mathbf{Bool}_P$, of all P-scaled Boolean algebras with morphisms defined as Boolean algebra homomorphisms $\varphi \colon A \to B$ with each $\varphi[A^{(p)}] \subseteq B^{(p)}$, is ω-accessible.

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- For each $p \in P$, $\mathrm{F}(X)^{(p)}$ is the ideal of $\mathrm{F}(X)$ generated by $\{\tilde{u} \mid p \leq \partial u\}$.
- Then $\mathbf{F}(X) \overset{\text{def}}{=} (\mathrm{F}(X), (\mathrm{F}(X)^{(p)} \mid p \in P))$ is a P-scaled Boolean algebra.

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The Boosting Lemma (W 2019)

Under quite general conditions, $\Gamma(\lambda) \in \operatorname{rng}\Phi$ as well.

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- It has order-dimension 3, thus one can take $\kappa = \aleph_{\alpha+2}$.

The functor V

- For any ring R, $\mathrm{V}(R)$ ("nonstable K_0-theory of R") is the set of Murray - von Neumann equivalence classes of all idempotent matrices over R, with addition defined by

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- V extends naturally to a functor, from the category of all rings with ring homomorphisms to the category **CMon** of all commutative monoids with monoid homomorphisms.

The diagrams $\vec{D}$ and $\vec{R}_{\Bbbk}$

- On $\mathbb{Z}^+$: $\boldsymbol{e}(x) \stackrel{\text{def}}{=} (x, x)$, $\boldsymbol{s}(x, y) \stackrel{\text{def}}{=} (y, x)$, $\boldsymbol{p}(x, y) \stackrel{\text{def}}{=} x + y$.

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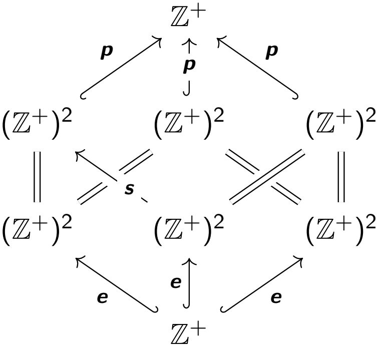

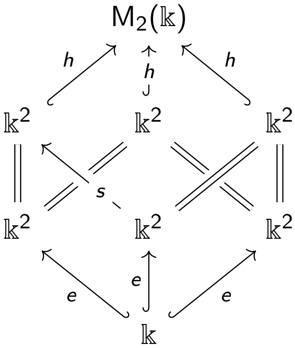

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- There is no commutative diagram $\vec{R}$, of V-semiprimitive rings, such that $\mathrm{V}(\vec{R}) \cong \mathrm{V}(\vec{R}_{\Bbbk})$ (W 2013).

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Then $\mathrm{V}(\mathcal{R})$ is anti-elementary.

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- Hence, the result above does not apply to $\mathbf{LocMat}_{\mathbb{k}}$ *a priori*.
- We still know that $V(\mathbf{LocMat}_{\mathbb{k}})$ is not closed under elementary extensions (thus not first-order; combine Elliott 1976 and W 1998).