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Large free sets versus k-ladders: combinatorial issues raised by "From lifting objects to lifting diagrams"

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Most of the results discussed here obtained with **Pierre Gillibert**.

June 2-6, 2010

An example with the $\mathrm{Con_c}$ functor

- Denote by $\mathrm{Con_c}\, A$ the $(\vee, 0)$-semilattice of all compact (i.e., finitely generated) congruences of an (universal) algebra A.

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- We would like to prove that every countable distributive $(\vee, 0)$-semilattice can be lifted in $\mathcal{V}$.

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- Let $A = \varinjlim_{n<\omega} A_n$. Then $\mathrm{Con_c}\, A \cong S$.

$\mathrm{Con_c}$ example, continued (2)

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- How to extend this to larger cardinalities?
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- ... Remaining question: what plays the role of $\omega = \{0, 1, 2, \dots\}$ (index poset of a direct system)?

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Observe that the 1-ladders are exactly the finite chains together with $\omega = \{0, 1, 2, \dots\}$.

Construction of a 2-ladder of cardinality $\aleph_1$

Idea of proof of (ii): $F = \bigcup_{\alpha<\omega_1} F_\alpha$. Start with $F_0 := \{0\}$. Suppose F_α constructed.

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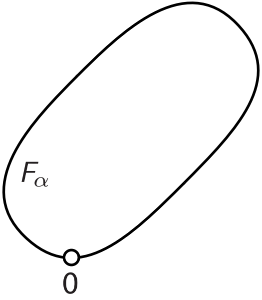

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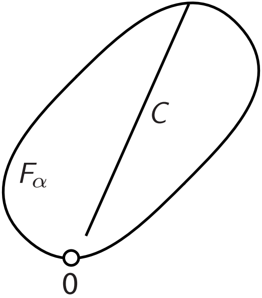

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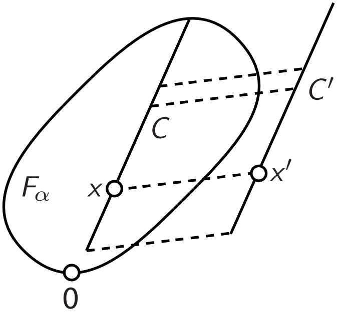

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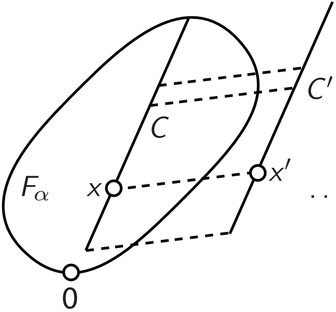

...and that's $F_{\alpha+1}$

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A few applications of 2-ladders

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In both results above, the $\aleph_1$ bound is optimal (different methods).

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Suppose that either $\mathrm{MA}(\aleph_1)$ holds or there exists a gap-1 morass. Then there exists a 3-ladder of cardinality $\aleph_2$.

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If there is no 3-ladder of cardinality $\aleph_2$, then ω_2 is inaccessible in $\mathbf{L}$.

On the other hand, 3-dimensional amalgamation fails in any variety with a nontrivial member.

Critical points

- Set $\operatorname{Con_c} \mathcal{V} := \{S \mid (\exists A \in \mathcal{V})(S \cong \operatorname{Con_c} A)\}$, for any class $\mathcal{V}$ of algebras.

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Let $\mathcal{A}$ and $\mathcal{B}$ be varieties of algebras, with $\mathcal{A}$ locally finite and $\mathcal{B}$ finitely generated congruence-distributive. If $\mathrm{Con_c}\,\mathcal{A} \not\subseteq \mathrm{Con_c}\,\mathcal{B}$, then $\mathbf{crit}(\mathcal{A};\mathcal{B}) < \aleph_\omega$.

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Theorem (Gillibert 2009)

Let $\mathcal{A}$ and $\mathcal{B}$ be varieties of lattices such that every simple member of $\mathcal{B}$ has a prime interval. If $\mathcal{A} \not\subseteq \mathcal{B}$ and $\mathcal{A} \not\subseteq \mathcal{B}^{\mathrm{dual}}$, then $\operatorname{Con_c} \mathcal{A} \not\subseteq \operatorname{Con_c} \mathcal{B}$ and $\mathbf{crit}(\mathcal{A};\mathcal{B}) \leq \aleph_2$. The bound $\aleph_2$ is optimal.

A question about critical points

Question

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Formally related to the existence problem of 3-ladders of cardinality $\aleph_2$... On the other hand, 3-dimensional amalgamation fails!

What to think?

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λ-lifters
$(\kappa, <\lambda) \rightsquigarrow P$
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What makes critical points small?

- Using the chain ω makes critical points $\geq \aleph_1$.

What makes critical points small?

Lifters, free sets, ladders

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- ...But what makes critical points small?
- Goto category theory...

General categorical settings

We are given categories $\mathcal{A}$, $\mathcal{B}$, $\mathcal{S}$ together with functors $\Phi\colon \mathcal{A} \to \mathcal{S}$ and $\Psi\colon \mathcal{B} \to \mathcal{S}$.

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Hence we need an assumption of the form "for many $A \in \mathcal{A}$, there exists $B \in \mathcal{B}$ such that $\Phi(A) \cong \Psi(B)$". Ask for $\Gamma\colon A \mapsto B$ to be a functor (at least on a large enough subcategory of $\mathcal{A}$).

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That is, if Ψ lifts many objects, then it lifts many (P-indexed) diagrams.

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For critical points, Φ and Ψ are both $\mathrm{Con_c}$.

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- an almost join-semilattice if it is a pseudo join-semilattice and $P \downarrow a$ is a join-semilattice $\forall a \in P$.

Examples (pseudo join-semilattice, etc.)

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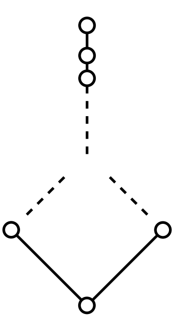

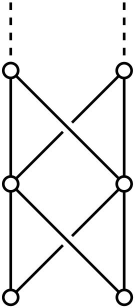

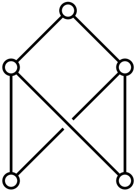

Norm-coverings, sharp ideals

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- A norm-covering of a poset P is a pseudo join-semilattice X, together with an isotone map $\partial\colon X \to P$.
- An ideal (i.e., a nonempty, upward directed, lower subset) $\mathbf{x}$ of X is sharp if $\{\partial x \mid x \in \mathbf{x}\}$ has a largest element, then denoted by $\partial\mathbf{x}$.

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- Then set $\mathbf{X}^{=} := \{\mathbf{x} \in \mathbf{X} \mid \partial\mathbf{x} \text{ not maximal}\}$, for every set $\mathbf{X}$ of sharp ideals.

λ-lifters

The posets P involved in the statement of CLL are those for which there exists a λ-lifter. Here, λ may be thought of as an upper bound for the sizes of the diagram data, while the cardinality of X may be thought as an upper bound for the sizes of the object data.

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(i) Every principal ideal of $\mathbf{X}^{=}$ has cardinality $< \mathrm{cf}(\lambda)$.

(ii) For every map $S\colon \mathbf{X}^{=} \to [X]^{<\lambda}$ there exists an isotone section σ of ∂ such that $S\sigma(a) \cap \sigma(b) \subseteq \sigma(a)$ for all $a < b$ in P.

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(iii) If $\lambda = \aleph_0$, then X is supported.

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(ii) For every isotone map $F\colon [\kappa]^{<\mathrm{cf}(\lambda)} \to [\kappa]^{<\lambda}$, there exists a one-to-one map $\sigma\colon P \rightarrowtail \kappa$ such that

$$(\forall a < b \text{ in } P)\big(F\sigma(P \downarrow a) \cap \sigma(P \downarrow b) \subseteq \sigma(P \downarrow a)\big)\,.$$

These posets are too heavy. . .

In particular, none of the following posets has a lifter:

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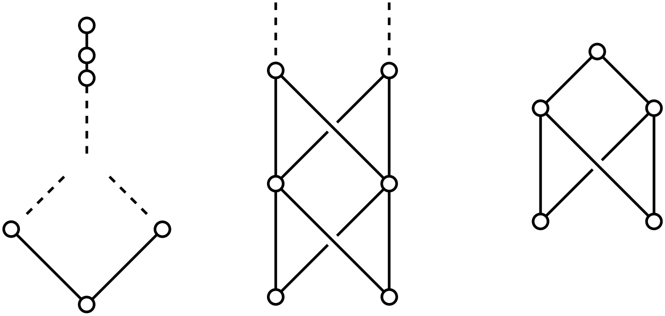

Well-foundedness and liftability

Theorem (Gillibert + W. 2010)

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Corollary

$(\omega+1)^{\mathrm{dual}}$ has no $(2^{\aleph_0})^{+}$-lifter.

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If "isotone" is removed from the assumptions above, then the answer to the corresponding question is easily seen to be negative.

A diagram counterexample with no associated object counterexample so far

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- In particular, this diagram cannot be lifted by lattices, majority algebras, groups, modules, etc.

A diagram counterexample with no associated object counterexample so far

The following uses 1998 results from Kearnes and Szendrei on commutator theory.

Theorem (Tůma and W., 2006)

There exists a diagram, indexed by a finite poset P, of finite Boolean semilattices and $(\vee, 0)$-embeddings, which cannot be lifted, with respect to the $\mathrm{Con_c}$ functor, in any variety satisfying a nontrivial congruence identity.

- In particular, this diagram cannot be lifted by lattices, majority algebras, groups, modules, etc.
- Nevertheless, it can be lifted by groupoids.

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This diagram has the following form:

$$
\begin{tikzcd}
& \mathfrak{P}(4) & \\
\mathfrak{P}(3) \arrow[ur, "\mathbf{u}_0"] & \mathfrak{P}(3) \arrow[u, "\mathbf{u}_1"] & \mathfrak{P}(3) \arrow[ul, "\mathbf{u}_2"'] \\
\mathfrak{P}(2) \arrow[u, "\mathbf{f}_0"] \arrow[ur, "\mathbf{f}_0"] \arrow[urr, "\mathbf{f}_0"] & \mathfrak{P}(2) \arrow[ul, "\mathbf{f}_1"] \arrow[u, "\mathbf{f}_1"] \arrow[ur, "\mathbf{f}_1"] & \mathfrak{P}(2) \arrow[ull, "\mathbf{f}_2"] \arrow[ul, "\mathbf{f}_2"] \arrow[u, "\mathbf{f}_2"'] \\
& \mathfrak{P}(1) \arrow[ul, "\mathbf{e}"] \arrow[u, "\mathbf{e}"] \arrow[ur, "\mathbf{e}"'] &
\end{tikzcd}
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- Its underlying poset does not have any lifter (because it is not an almost join-semilattice).

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- Thus we cannot apply CLL (or its known extensions) to it.
- In particular, it is still unknown whether every distributive $(\vee, 0)$-semilattice is isomorphic to $\mathrm{Con_c}\, M$ for some majority algebra M.

The $(\kappa, <\lambda) \rightsquigarrow P$ notation

The following is a more user-friendly variant of the definition of a lifter.

Definition (Gillibert and W., 2008)

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For infinite cardinals κ, λ and a poset P, let $(\kappa, {<}\lambda) \rightsquigarrow P$ hold if for every mapping $F \colon \mathfrak{P}(\kappa) \to [\kappa]^{<\lambda}$ there exists a one-to-one map $\sigma \colon P \rightarrowtail \kappa$ such that

$$(\forall x < y \text{ in } P)\big(F\sigma(P \downarrow x) \cap \sigma(P \downarrow y) \subseteq \sigma(P \downarrow x)\big)\,.$$

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In case P is lower finite, if is sufficient to replace $P \downarrow z$ by $\mathrm{J}(P) \downarrow z$ in the statement above (convenient for verifications on specific posets).

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Theorem (Gillibert and W., 2008)

Let P be a finite poset and let λ be an infinite cardinal. Then P has a λ-lifter iff P is a disjoint union of almost join-semilattices with zero.

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- The smallest possible cardinality of a λ-lifter of a finite poset P can be estimated precisely, *via* the $(\kappa, {<}\lambda) \leadsto P$ notation.
- A convenient way to do this is to introduce the Kuratowski index.

The Kuratowski index of a finite poset

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The Kuratowski index of a finite poset P, denoted by $\mathrm{kur}(P)$, is defined as 0 if P is an antichain, and, otherwise, as the least $n > 0$ such that $(\lambda^{+(n-1)}, {<}\lambda) \rightsquigarrow P$ for each infinite cardinal λ.

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Theorem (Gillibert + W., 2008)

The Kuratowski index of P always exists, and $\mathrm{kur}(P) \leq \dim(P) \leq \mathrm{width}\,\mathrm{J}(P) \leq \mathrm{card}\,\mathrm{J}(P)$. Furthermore, if P is a join-semilattice, then $\mathrm{breadth}(P) \leq \mathrm{kur}(P)$.

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Theorem (Gillibert + W., 2008)

For every infinite cardinal λ, every nontrivial finite almost join-semilattice P with zero has a λ-lifter $(X, \mathbf{X})$ with $\mathrm{card}\,X = \mathrm{card}\,\mathbf{X} = \lambda^{+(\mathrm{kur}(P)-1)}$.

Sometimes, $\mathrm{kur}(P)$ is easy to calculate:

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- $\mathrm{kur}(T) = 1$ whenever T is a nontrivial finite tree.

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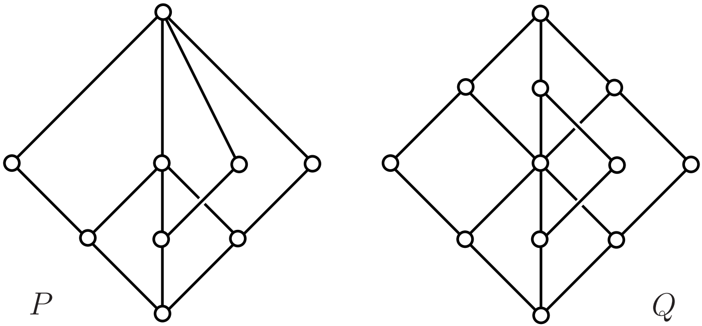
- In particular, $\mathrm{kur}(\mathbf{2}^n) = n$ (also follows from Kuratowski's Free Set Theorem).
- Warning: the $P \mapsto \mathrm{kur}(P)$ function is not absolute (in the set-theoretical sense).

. . . sometimes, not so easy:

- Consider the finite lattices P and Q represented below.



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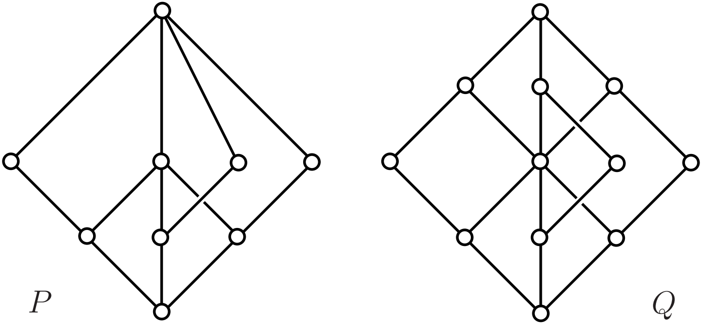

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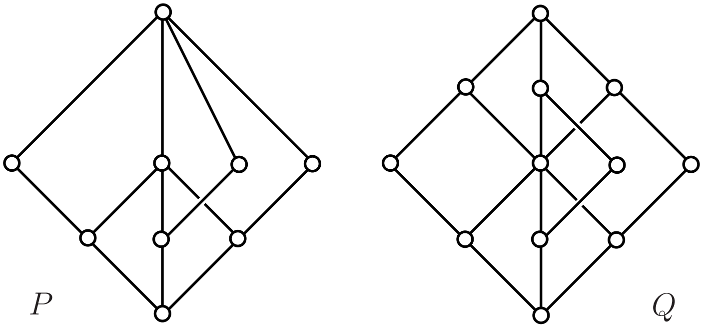

- $\mathrm{breadth}(P) = \mathrm{breadth}(Q) = 2$ while $\dim(P) = \dim(Q) = 3$.
- Thus $2 \leq \mathrm{kur}(P) \leq \mathrm{kur}(Q) \leq 3$.

We don't know more. For example, $\mathrm{kur}(P) = 2$ would mean the following:

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For every infinite cardinal λ and every $F \colon [\lambda^+]^{<\omega} \to [\lambda^+]^{<\lambda}$, there are distinct $\xi_0, \xi_1, \xi_2, \eta_0, \eta_1, \eta_2 < \lambda$ such that $\xi_i \notin F(\{\xi_j, \eta_j\})$, $\eta_i \notin F(\{\xi_j, \eta_j\})$, and $\eta_i \notin F(\{\xi_0, \xi_1, \xi_2\})$ for all $i \neq j$ in $\{0, 1, 2\}$.

Truncated m-cubes

For $1 \leq r \leq m$, we define the truncated m-dimensional cube

$$\mathrm{B}_m({\leqslant}r) := \{X \in \mathfrak{P}(m) \mid \text{either } \operatorname{card} X \leq r \text{ or } X = m\},$$

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Proposition (Gillibert + W., 2008)

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Proposition (Gillibert + W., 2008)

Let r and m be integers with $1 \leq r < m$ and let κ and λ be infinite cardinals. Then $(\kappa, {<}\lambda) \leadsto \mathrm{B}_m(\leqslant r)$ iff $(\kappa, r, \lambda) \to m$, that is, for each $F \colon [\kappa]^r \to [\kappa]^{<\lambda}$, there exists $H \in [\kappa]^m$ such that $F(X) \cap H \subseteq X$ for each $X \in [H]^r$ (we say that H is free with respect to F).

Kuratowski indexes of near-bottom truncated cubes

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(i) If GCH holds, then $(\lambda^{+r}, r, \lambda) \to \lambda^{+}$ for every infinite cardinal λ and every integer $r \geq 1$ (Erdős *et al.*, 1984).

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(iii) $(\lambda^{+3}, 3, \lambda) \to m$ for every integer $m > 0$ (Hajnal, 1984). Hence $\operatorname{kur} \mathrm{B}_m(\leqslant 3) = 4$ for all $m \geq 4$.

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Item (ii) above has been used by Ploščica and Gillibert to evaluate various critical points between finitely generated modular lattice varieties.

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Higher truncated cubes

Set $t_0 := 5$, $t_1 := 7$, and, for each $n > 0$, let $t_{n+1} \to (t_n, 7)^5$ (they exist, due to Ramsey's Theorem).

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- There exists a generic extension of the universe in which $(\aleph_4, 4, \aleph_0) \not\to t_4$.
- Hence $\operatorname{kur} \mathrm{B}_{t_4}(\leqslant 4) = 5$ in any universe with GCH, while $\operatorname{kur} \mathrm{B}_{t_4}(\leqslant 4) \geq 6$ in some generic extension (hence $P \mapsto \operatorname{kur}(P)$ is not absolute).

A new lower bound (size of a free set)

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breadth $\mathrm{B}_{n+2}(\leqslant n) = \dim \mathrm{B}_{n+2}(\leqslant n) = n+1$, for all $n > 0$.

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- Next open question: $(\aleph_4, 4, \aleph_0) \to 7$. That is, does $\operatorname{kur} \mathrm{B}_7(\leqslant 4) = 5$?
- The unreachable upper limit: remember that $(\aleph_4, 4, \aleph_0) \not\to t_4$ in some generic extension.

Using further order-dimension estimates

- Remember the estimate $\mathrm{kur}(P) \leq \dim(P)$.

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- $(\aleph_7, 4, \aleph_0) \rightarrow 10$ (using Dushnik's estimate).

Using further order-dimension estimates

- Remember the estimate $\mathrm{kur}(P) \leq \dim(P)$.
- Estimating the order-dimension $\mathrm{kur}\,\mathrm{B}_m(\leqslant r)$ has been an object of intensive study (Dushnik, Füredi + Kahn, Kierstead, Hajnal, Spencer, and others).
- Using the various estimates available, we obtain, for example:

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- $(\aleph_{109}, 4, \aleph_0) \to 257$ (using Füredi and Kahn's estimate).

Using further order-dimension estimates

The Con_c functor
k-ladders
Critical points
From lifting objects to lifting diagrams
λ-lifters
$(\kappa, <\lambda) \rightsquigarrow P$
Calculating Kuratowski indexes of finite posets

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- $(\aleph_{109}, 4, \aleph_0) \rightarrow 257$ (using Füredi and Kahn's estimate).
- $(\aleph_{210}, 4, \aleph_0) \rightarrow 32{,}768$ (using Hajnal and Spencer's estimate).

For general n and r, $(\aleph_n, r, \aleph_0) \to E(n, r)$ with

$$\lg \lg E(n, r) \sim \frac{n}{r 2^r \log 2} \text{ as } n \gg r \gg 0$$

(again by using Hajnal and Spencer's estimate).