Type monoids of Boolean inverse semigroups

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Idempotents of $\mathrm{Inv}(\Omega, G)$: they are the identities on all subsets of Ω. They form a Boolean lattice.

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Large class of Boolean inverse semigroups: all $\operatorname{Inv}(\mathcal{B}, G)$.

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Those S in which $a \wedge b$ always exists are called inverse meet-semigroups.

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A semigroup homomorphism $f\colon S \to T$, between Boolean inverse semigroups, is additive if $x \perp_S y$ implies that $f(x) \perp_T f(y)$ and $f(x \oplus y) = f(x) \oplus f(y)$.

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Both $x \mathbin{\oslash} y$ and $x \mathbin{\triangledown} y$ are always defined.

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- Therefore, the variety of all biases is congruence-permutable. (Note: it is not congruence-distributive.)
- Hence, Boolean inverse semigroups are much closer to rings than to semigroups.

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- The representation above is called the regular representation of S.

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- The type monoid of S, denoted by Typ S, is the universal monoid of the partial commutative monoid $\operatorname{Int} S$.

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- Then the type monoid of $\operatorname{Inv}(\mathcal{B}, G)$ is isomorphic to $\mathbb{Z}^+\langle\mathcal{B}\rangle /\!\!/ G$.

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- Can be solved by representing $\mathcal{B}$ as generalized Boolean lattice of subsets of some set Ω, then duplicating Ω. This leaves enough room to extend f_x.

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- How about the converse?

Dobbertin's V-measures

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Let M be a countable, conical refinement monoid and let $\boldsymbol{e} \in M$.

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Let M be a countable, conical refinement monoid and let $\boldsymbol{e} \in M$. Then there are a countable Boolean algebra B and a finitely additive measure $\mu\colon B \to M$ such that $\mu(1) = \boldsymbol{e}$, $\mu^{-1}\{0\} = \{0\}$, and whenever $\mu(c) = \boldsymbol{a} + \boldsymbol{b}$, there exists a decomposition $c = a \oplus b$ in B such that $\mu(a) = \boldsymbol{a}$ and $\mu(b) = \boldsymbol{b}$.

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For card $M = \aleph_1$, uniqueness is lost. If card $M \geq \aleph_2$, then existence is lost (W 1998).

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Proof of Dobbertin's Theorem: essentially back-and-forth.

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- S is a Boolean inverse semigroup, with idempotents $\overline{a} = \mathrm{id}_{B\downarrow a}$ where $a \in B$.

Measurability of countable CRMs (cont'd)

- Because of the uniqueness statement in Dobbertin's Theorem, for any $a, b \in B$, if $\mu(a) = \mu(b)$, there is $f \in S$ (usually not unique) such that $f(a) = b$.

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- Adding the condition $a_0 \neq \perp$ (i.e., each $a_i \in G$) yields a Boolean subring B of $\overline{B}$.
- The dimension monoid $\operatorname{Dim} G$ of the (distributive) lattice $(G, \vee, \wedge)$ is isomorphic to the monoid $\mathbb{Z}^+\langle B\rangle$ of all nonnegative linear combinations of members of B, with $\oplus$ in B turned to $+$ in $\mathbb{Z}^+\langle B\rangle$.

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(where $a_0 \leq a_1 \leq \cdots \leq a_{2n}$ in G).

- Moreover, $\forall a \in G$, the translation $x \mapsto x + a$ "extends" to an automorphism τ_a of B. So $\tau_a(y \smallsetminus x) = (a + y) \smallsetminus (a + x)$, $\forall x \leq y$ in G.

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- Getting "locally matricial" in arbitrary cardinality: hopeless for arbitrary dimension groups (counterexamples of size $\aleph_2$), but still open for abelian ℓ-groups.

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- If $X \subseteq S$ generates S as a bias, then it also generates $K\langle S\rangle$ as an involutary subring.

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Definition

For a unital ring K and a BIS S, $K\langle S\rangle$ is the K-algebra defined by generators S and relations $\lambda s = s\lambda$, $1s = s$, $z = x + y$ (within $K\langle S\rangle$) whenever $z = x \oplus y$ (within S).

- For S a Boolean inverse meet-semigroup, $K\langle S\rangle$ is isomorphic to Steinberg's $K\,\mathscr{U}_{\mathrm{T}}(S)$ (*étale groupoid algebra* of $\mathscr{U}_{\mathrm{T}}(S)$), where $\mathscr{U}_{\mathrm{T}}(S)$ is called there the *universal additive groupoid* of S. Steinberg's construction extends to *Hausdorff inverse semigroups* (not necessarily Boolean).
- If K is an involutary ring, then $K\langle S\rangle$ is an involutary K-algebra (set $(\lambda s)^* = \lambda^* s^{-1}$).
- If $X \subseteq S$ generates S as a bias, then it also generates $K\langle S\rangle$ as an involutary subring.
- The construction $K\langle S\rangle$ extends known constructions, such as *Leavitt path algebras*.

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- Can, in certain conditions, be extended to involutary semirings.
- Yields a workable definition of the tensor product $S \otimes T$ of two BISs S and T, which is still a BIS and has $\mathrm{Idp}(S \otimes T) \cong (\mathrm{Idp}\, S) \otimes (\mathrm{Idp}\, T)$, $\mathrm{U}_{\mathrm{mon}}(S \otimes T) \cong \mathrm{U}_{\mathrm{mon}}(S) \otimes \mathrm{U}_{\mathrm{mon}}(T)$, and $\mathrm{Typ}(S \otimes T) \cong \mathrm{Typ}(S) \otimes \mathrm{Typ}(T)$.

Embedding properties of $K\langle S\rangle$

- For S a sub-BIS ($\leftrightharpoons$ sub-bias) of T, the canonical map $K\langle S\rangle \to K\langle T\rangle$ may not be one-to-one.

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- Has to do with so-called transfer properties in lattice theory (getting from $K \hookrightarrow \operatorname{Id} L$ to $K \hookrightarrow L$).

The canonical map Typ $S \to \mathrm{V}(K\langle S\rangle)$

- For idempotent matrices a and b from a ring R, let $a \sim b$ hold if $\exists x, y$, $a = xy$ and $b = yx$ (*Murray - von Neumann equivalence*).

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Let S be a BIS and let K be a unital ring. Then there is a unique monoid homomorphism $\boldsymbol{f}\colon \mathrm{Typ}\, S \to \mathrm{V}(K\langle S\rangle)$ such that $\boldsymbol{f}(x/\mathscr{D}) = [x]_{K\langle S\rangle}\ \forall x \in S$.

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- Question: does $\operatorname{Typ} S \cong \mathrm{V}(\mathbb{Z}\langle S\rangle)$?