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A counterexample to the Congruence Lattice Problem

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Background: algebraic lattices

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- An element a in a lattice $(A, \vee, \wedge)$ is compact, if $a \leq \bigvee X$ (for $X \subseteq A$) implies that there exists a finite $Y \subseteq X$ such that $a \leq \bigvee Y$.

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- A lattice A is algebraic, if it is complete and every element of A is a join (=supremum) of compact elements.

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- An element a in a lattice $(A, \vee, \wedge)$ is compact, if $a \leq \bigvee X$ (for $X \subseteq A$) implies that there exists a finite $Y \subseteq X$ such that $a \leq \bigvee Y$.
- A lattice A is algebraic, if it is complete and every element of A is a join (=supremum) of compact elements.
- For an algebra U, the lattice Con U of all congruences of U (with $\subseteq$) is algebraic (Birkhoff and Frink, 1948).

Background: the Grätzer-Schmidt Theorem

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Theorem (Grätzer and Schmidt, 1963)

Every algebraic lattice A is isomorphic to Con U, for some algebra U.

Background: the Grätzer-Schmidt Theorem

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The algebra U constructed above may have many operations. This is unavoidable in general (Freese, Lampe, and Taylor, 1979),

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Theorem (Grätzer and Schmidt, 1963)

Every algebraic lattice A is isomorphic to Con U, for some algebra U.

The algebra U constructed above may have many operations. This is unavoidable in general (Freese, Lampe, and Taylor, 1979), but U can be taken a groupoid in case the lattice A has a compact unit (Lampe, 1982).

Background: compact congruences

- The compact congruences of an algebra U (that is, the compact elements of $\operatorname{Con} U$) are the finite joins of principal congruences $\operatorname{con}_U(x, y)$ (i.e., the least congruence of U that identifies x and y), where $x, y \in U$.

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- The collection $\mathrm{Con_c}\, U$ of all compact congruences of U (with $\subseteq$) is a $(\vee, 0)$-semilattice.

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- The collection $\operatorname{Con}_c U$ of all compact congruences of U (with $\subseteq$) is a $(\vee, 0)$-semilattice.
- Any homomorphism $f\colon U \to V$ of algebras of the same signature gives rise to a $(\vee, 0)$-homomorphism $\operatorname{Con}_c f\colon \operatorname{Con}_c U \to \operatorname{Con}_c V$, defined by the rule

$$(\operatorname{Con}_c f)(\alpha) = \bigvee \Big(\operatorname{con}_V(f(x), f(y)) \mid (x, y) \in \alpha \Big).$$

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$$(\operatorname{Con_c} f)(\alpha) = \bigvee \Big(\operatorname{con}_V(f(x), f(y)) \mid (x, y) \in \alpha \Big).$$

- Hence $U \mapsto \operatorname{Con_c} U$, $f \mapsto \operatorname{Con_c} f$ defines a functor from algebras of a same signature with their homomorphisms to $(\vee, 0)$-semilattices and $(\vee, 0)$-homomorphisms. This functor preserves direct limits.

Funayama and Nakayama's Theorem

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Con L is distributive, for any lattice L.

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The proof uses a majority operation on L, for example,

$$\mathbf{m}(x, y, z) = (x \wedge y) \vee (x \wedge z) \vee (y \wedge z) \quad (x, y, z \in L).$$

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(where majority operation just means that $\mathbf{m}(x, x, y) = \mathbf{m}(x, y, x) = \mathbf{m}(y, x, x) = x$.)

Dilworth's finite converse

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At about that time Dilworth discovered the following converse:

Theorem (Dilworth, ∼1940s, unpublished)

Every finite distributive lattice is isomorphic to Con L, for some finite lattice L.

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First published proof of the theorem above due to Grätzer and Schmidt (1963). Their lattice L is sectionally complemented,

Dilworth's finite converse

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Every finite distributive lattice is isomorphic to Con L, for some finite lattice L.

First published proof of the theorem above due to Grätzer and Schmidt (1963). Their lattice L is sectionally complemented, that is, for each $x \leq y$ in L, there exists $z \in L$ such that $x \wedge z = 0$ and $x \vee z = y$ (*abbreviation*: $y = x \oplus z$).

The Congruence Lattice Problem

The general problem is traditionally attributed to Dilworth:

The Congruence Lattice Problem, CLP

Is every distributive algebraic lattice isomorphic to Con L, for some lattice L?

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Its first printed appearance was in Grätzer and Schmidt, 1962, but it was obviously known earlier.

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Its first printed appearance was in Grätzer and Schmidt, 1962, but it was obviously known earlier. For more history on this problem, see G. Grätzer, "Two problems that shaped a century of lattice theory", Notices Amer. Math. Soc. **54**, no. 6 (June/July 2007), 696–707.

Algebraic lattices and $(\vee, 0)$-semilattices

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- $\mathcal{A} :=$ category of all algebraic lattices with compactness-preserving complete join-homomorphisms (but not necessarily meet-homomorphisms).

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- With an algebraic lattice A, associate the $(\vee, 0)$-semilattice $\operatorname{Komp} A$ of all compact elements of A.

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- Extends naturally to a functor $\operatorname{Komp}\colon \mathcal{A} \to \mathcal{S}$.

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- With an algebraic lattice A, associate the $(\vee, 0)$-semilattice $\operatorname{Komp} A$ of all compact elements of A.
- Extends naturally to a functor $\operatorname{Komp}\colon \mathcal{A} \to \mathcal{S}$.
- With a $(\vee, 0)$-semilattice S, associate the lattice $\operatorname{Id} S$ of all ideals of S, that is, all lower subsets of S that are also $(\vee, 0)$-subsemilattices of S.

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The category equivalence between $\mathcal{A}$ and $\mathcal{S}$

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Theorem (folklore, ∼1950s)

The pair of functors $(\mathrm{Komp}, \mathrm{Id})$ extends naturally to a category equivalence between $\mathcal{A}$ (algebraic lattices) and $\mathcal{S}$ ($(\vee, 0)$-semilattices).

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So, algebraic lattices are "the same" as $(\vee, 0)$-semilattices. The lattice $\mathrm{Con}\, U$ (for an algebra U), which is an algebraic lattice, corresponds to the $(\vee, 0)$-semilattice $\mathrm{Con_c}\, U$. Most problems related to CLP are more conveniently formulated in the language of $(\vee, 0)$-semilattices.

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Definition

A $(\vee, 0)$-semilattice S is distributive, if for all $a, b, c \in S$ such that $c \leq a \vee b$, there are $x \leq a$ and $y \leq b$ such that $c = x \vee y$.

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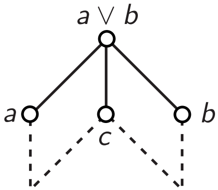

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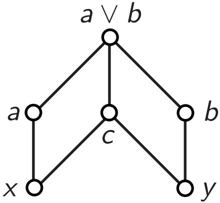

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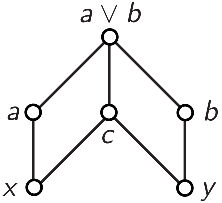

Equivalently, the ideal lattice $\mathrm{Id}\, S$ is a distributive lattice.

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Semilattice formulation of CLP

Is every distributive $(\vee, 0)$-semilattice S representable, that is, isomorphic to $\mathrm{Con_c}\, L$, for some lattice L?

Semilattice formulation of CLP

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Many partial positive results are known.

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The main negative result

Theorem (FW, 2005)

There exists a distributive $(\vee, 0, 1)$-semilattice S that is not isomorphic to $\mathrm{Con_c}\, L$, for any lattice L.

The main negative result

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The original proof has $|S| = \aleph_{\omega+1}$.

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Now let's outline the proof of the theorem above.

The Kuratowski Free Set Theorem

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For any set Ω and any natural number n, we put

$$[\Omega]^n = \{X \subseteq \Omega \mid |X| = n\},$$
$$[\Omega]^{<\omega} = \{X \subseteq \Omega \mid X \text{ is finite}\}.$$

The Kuratowski Free Set Theorem

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Kuratowski's Free Set Theorem (1951)

Let n be a natural number and let Ω be a set. Then $|\Omega| \geqslant \aleph_n$ iff any mapping $\Phi\colon [\Omega]^n \to [\Omega]^{<\omega}$ has a $(n+1)$-element free set.

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The following definition is a modification of E. T. Schmidt's original 1968 definition of a weakly distributive homomorphism.

Definition

For join-semilattices S and T, a join-homomorphism $\mu\colon S \to T$ is weakly distributive at $\mathbf{x} \in S$, if for all $\mathbf{y}_0, \mathbf{y}_1 \in T$ such that $\mu(\mathbf{x}) \leq \mathbf{y}_0 \vee \mathbf{y}_1$, there are $\mathbf{x}_0, \mathbf{x}_1 \in S$ such that $\mathbf{x} \leq \mathbf{x}_0 \vee \mathbf{x}_1$ and $\mu(\mathbf{x}_i) \leq \mathbf{y}_i$, for all $i < 2$.

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Fundamental examples:

- For any reduct A of an algebra B (in any signature), the canonical map $\operatorname{Con_c} A \to \operatorname{Con_c} B$ is a weakly distributive $(\vee, 0)$-homomorphism.

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Fundamental examples:

- For any reduct A of an algebra B (in any signature), the canonical map $\operatorname{Con_c} A \to \operatorname{Con_c} B$ is a weakly distributive $(\vee, 0)$-homomorphism.
- For any convex sublattice K of a lattice L, the canonical map $\operatorname{Con_c} K \to \operatorname{Con_c} L$ is a weakly distributive $(\vee, 0)$-homomorphism.

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We need $(\vee, 0, 1)$-semilattices with lots of elements $\mathbf{a}$, $\mathbf{b}$ such that $\mathbf{a} \vee \mathbf{b} = 1$. The most natural choice is to use the free objects with collections of such pairs.

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This extends naturally to a functor $\mathcal{L}$ from **Set** to the category of all $(\vee, 0, 1)$-semilattices with $(\vee, 0, 1)$-homomorphisms. The functor $\mathcal{L}$ preserves direct limits.

Concrete representation of $\mathcal{L}(\Omega)$: it consists of all pairs $(X, Y) \in \mathfrak{P}(\Omega) \times \mathfrak{P}(\Omega)$ such that

$$\text{either } X \text{ and } Y \text{ are finite and disjoint, or } X = Y = \Omega;$$

then $\mathbf{a}_0^\xi = (\{\xi\}, \varnothing)$ and $\mathbf{a}_1^\xi = (\varnothing, \{\xi\})$.

$\mathcal{L}(\aleph_{\omega+1})$ could almost have worked, except that...

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For a $(\vee, 0)$-semilattice S, we put

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and we denote by $\mathcal{R}(S)$ the $(\vee, 0)$-semilattice freely generated by S and elements $\bowtie(\mathbf{a}, \mathbf{b}, \mathbf{c})$, for $(\mathbf{a}, \mathbf{b}, \mathbf{c}) \in \mathcal{C}(S)$, subjected to the relations

$$\bowtie(\mathbf{a}, \mathbf{b}, \mathbf{c}) \leq \mathbf{a},$$
$$\mathbf{c} = \bowtie(\mathbf{a}, \mathbf{b}, \mathbf{c}) \vee \bowtie(\mathbf{b}, \mathbf{a}, \mathbf{c}).$$

A concrete realization of $\mathcal{R}(S)$

A finite subset $\mathbf{x}$ of $\mathcal{C}(S)$ is reduced, if it satisfies the three conditions below:

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A finite subset $\mathbf{x}$ of $\mathcal{C}(S)$ is reduced, if it satisfies the three conditions below:

- $\mathbf{x}$ contains exactly one diagonal triple, that is, a triple of the form $(\mathbf{u}, \mathbf{u}, \mathbf{u})$. We put $\mathbf{u} = \pi(\mathbf{x})$.

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- $(\mathbf{u}, \mathbf{v}, \mathbf{w}) \in \mathbf{x}$ non-diagonal implies that $\mathbf{u}, \mathbf{v}, \mathbf{w} \not\leq \pi(\mathbf{x})$.

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The ordering on $\mathcal{R}(S)$ is defined by

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The ordering on $\mathcal{R}(S)$ is defined by

$$\mathbf{x} \leq \mathbf{y} \iff \forall (\mathbf{u}, \mathbf{v}, \mathbf{w}) \in \mathbf{x} \setminus \mathbf{y}, \quad \text{either } \mathbf{u} \leq \pi(\mathbf{y}) \text{ or } \mathbf{w} \leq \pi(\mathbf{y}).$$

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Theorem (Ploščica and Tůma 1997)

The two definition of $\mathcal{R}(S)$ presented above are equivalent.

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The two definition of $\mathcal{R}(S)$ presented above are equivalent.

- The canonical embedding from S into $\mathcal{R}(S)$ is the map $\mathbf{u} \mapsto (\mathbf{u}, \mathbf{u}, \mathbf{u})$.

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- We identify $\mathbf{u} \in S$ with $(\mathbf{u}, \mathbf{u}, \mathbf{u}) \in \mathcal{R}(S)$.

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We put $\mathcal{D}(S) = \bigcup_{n<\omega} \mathcal{R}^n(S)$, for each $(\vee, 0)$-semilattice S.

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Each "refinement problem" $\mathbf{c} \leq \mathbf{a} \vee \mathbf{b}$ in S has a solution in $\mathcal{R}(S)$. Hence, $\mathcal{D}(S)$ is a distributive $(\vee, 0)$-semilattice (in which S is cofinal).

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Preservation results for $\mathcal{R}$ and $\mathcal{D}$

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Lemma

For every collection $(S_i \mid i \in I)$ of $(\vee, 0)$-subsemilattices of a $(\vee, 0)$-semilattice S,

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For every collection $(S_i \mid i \in I)$ of $(\vee, 0)$-subsemilattices of a $(\vee, 0)$-semilattice S,

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Preservation results for $\mathcal{R}$ and $\mathcal{D}$

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Definition

Set $\mathcal{G} = \mathcal{D} \circ \mathcal{L}$.

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The support of $\mathbf{x} \in \mathcal{G}(\Omega)$ is the least $X \subseteq \Omega$ such that $\mathbf{x} \in \mathcal{G}(X)$. We denote it by $\operatorname{supp}(\mathbf{x})$.

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By the Lemma above, $\operatorname{supp}(\mathbf{x})$ is indeed defined, and it is a finite subset of Ω.

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By working with the "concrete realization of $\mathcal{R}(S)$", we can prove

The Evaporation Lemma

Let α, β, and δ be distinct elements in a set Ω, let $i, j < 2$, let $\mathbf{x} \in \mathcal{G}(\Omega \setminus \{\beta\})$, $\mathbf{y} \in \mathcal{G}(\Omega \setminus \{\alpha\})$, and $\mathbf{z} \in \mathcal{G}(\Omega \setminus \{\delta\})$. Then

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(join evaluated in the ideal lattice of $\mathcal{G}(\Omega)$.)

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- We are working in any algebra L with a congruence-compatible structure of a join-semilattice, say $(L, \vee)$. So every congruence of L is a $\vee$-congruence.

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- We also denote by $\mathrm{Con}_{\mathrm{c}}^{U} L$ the $(\vee, 0)$-subsemilattice of $\mathrm{Con}_{\mathrm{c}} L$ generated by all congruences $\mathrm{con}(u, v)$, where $(u, v) \in U \times U$.

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- Finally, we put $\varepsilon(n) = n \mod 2$, for any integer n.

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- Finally, we put $\varepsilon(n) = n \mod 2$, for any integer n. (That is, 0 if n is even, 1 if n is odd).

Statement of the Erosion Lemma

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The statement below is a slightly less general form of the original Erosion Lemma.

The Erosion Lemma

Let $x_0, x_1 \in L$ and let $Z = \{z_0, z_1, \ldots, z_n\} \subseteq L$ with $z_0 \leq x_0, x_1$ and $z_n = 1$ (largest element of L).

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$$\mathbf{a}_j = \bigvee \left(\operatorname{con}(z_i, z_{i+1}) \mid i \in \varepsilon^{-1}\{j\}\right) \quad (\forall j < 2).$$

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Then there are congruences $\mathbf{u}_j \in \mathrm{Con}_{\mathrm{c}}^{\{x_j\} \vee Z} L$, for $j < 2$, such that

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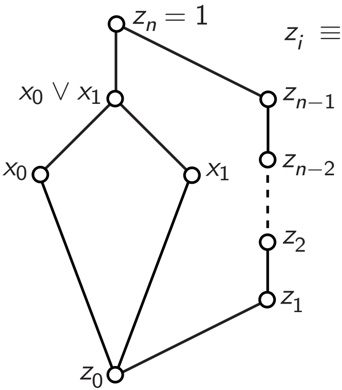
Then there are congruences $\mathbf{u}_j \in \operatorname{Con}_{\mathrm{c}}^{\{x_j\} \vee Z} L$, for $j < 2$, such that

$$x_0 \vee x_1 \equiv 1 \pmod{\mathbf{u}_0 \vee \mathbf{u}_1},$$
$$\mathbf{u}_j \subseteq \mathbf{a}_j \cap \operatorname{con}(x_j, 1) \quad (\forall j < 2).$$

Illustrating the Erosion Lemma



$$z_i \equiv z_{i+1} \pmod{\mathbf{a}_{\varepsilon(i)}}$$
$$(\varepsilon(i) = i \mod 2)$$

$$x_j \equiv 1 \pmod{\mathbf{x}_j}$$
$$\mathbf{u}_j \subseteq \mathbf{a}_j \cap \mathbf{x}_j$$
$$x_0 \vee x_1 \equiv 1 \pmod{\mathbf{u}_0 \vee \mathbf{u}_1}$$

with $\mathbf{u}_j \in \mathrm{Con}_{\mathrm{c}}^{\{x_j\} \vee Z} L$.

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Put $\mathbf{v}_i = \mathrm{con}(z_i \vee x_{\varepsilon(i)}, z_{i+1} \vee x_{\varepsilon(i)})$ $(\forall i < n)$.

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Put $\mathbf{v}_i = \mathrm{con}(z_i \vee x_{\varepsilon(i)}, z_{i+1} \vee x_{\varepsilon(i)})$ $(\forall i < n)$. So $\mathbf{v}_i \in \mathrm{Con}_{\mathrm{c}}^{\{x_{\varepsilon(i)}\} \vee Z} L$ and $\mathbf{v}_i \leq \mathbf{a}_{\varepsilon(i)}$.

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Put $\mathbf{v}_i = \operatorname{con}(z_i \vee x_{\varepsilon(i)}, z_{i+1} \vee x_{\varepsilon(i)})$ $(\forall i < n)$. So $\mathbf{v}_i \in \operatorname{Con}_{\mathrm{c}}^{\{x_{\varepsilon(i)}\} \vee Z} L$ and $\mathbf{v}_i \leq \mathbf{a}_{\varepsilon(i)}$. Put $\theta_i = \operatorname{con}(x_{\varepsilon(i)}, 1)$.

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Put $\mathbf{v}_i = \operatorname{con}(z_i \vee x_{\varepsilon(i)}, z_{i+1} \vee x_{\varepsilon(i)})$ $(\forall i < n)$. So $\mathbf{v}_i \in \operatorname{Con}_{\mathrm{c}}^{\{x_{\varepsilon(i)}\} \vee Z} L$ and $\mathbf{v}_i \leq \mathbf{a}_{\varepsilon(i)}$. Put $\theta_i = \operatorname{con}(x_{\varepsilon(i)}, 1)$. As $z_i \vee z_n = z_{i+1} \vee z_n (= 1)$ and $1 \equiv x_{\varepsilon(i)} \pmod{\theta_i}$,

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Put $\mathbf{v}_i = \mathrm{con}(z_i \vee x_{\varepsilon(i)}, z_{i+1} \vee x_{\varepsilon(i)})$ $(\forall i < n)$. So $\mathbf{v}_i \in \mathrm{Con}_{\mathrm{c}}^{\{x_{\varepsilon(i)}\}\vee Z} L$ and $\mathbf{v}_i \leq \mathbf{a}_{\varepsilon(i)}$. Put $\theta_i = \mathrm{con}(x_{\varepsilon(i)}, 1)$. As $z_i \vee z_n = z_{i+1} \vee z_n (= 1)$ and $1 \equiv x_{\varepsilon(i)} \pmod{\theta_i}$, we obtain that $z_i \vee x_{\varepsilon(i)} \equiv z_{i+1} \vee x_{\varepsilon(i)} \pmod{\theta_i}$, that is,

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$$\mathbf{v}_i \subseteq \operatorname{con}(x_{\varepsilon(i)}, 1).$$

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Now we put

$$\mathbf{u}_j = \bigvee (\mathbf{v}_i \mid i \in \varepsilon^{-1}\{j\}) \quad (\forall j < 2).$$

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So $\mathbf{u}_j \in \mathrm{Con}_{\mathrm{c}}^{\{x_j\} \vee Z} L$

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So $\mathbf{u}_j \in \operatorname{Con}_{\mathrm{c}}^{\{x_j\}\vee Z} L$ and $\mathbf{u}_j \subseteq \mathbf{a}_j \cap \operatorname{con}(x_j, 1)$.

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So $\mathbf{u}_j \in \operatorname{Con}_{\mathrm{c}}^{\{x_j\} \vee Z} L$ and $\mathbf{u}_j \subseteq \mathbf{a}_j \cap \operatorname{con}(x_j, 1)$. Finally, from $z_i \vee x_{\varepsilon(i)} \equiv z_{i+1} \vee x_{\varepsilon(i)} \pmod{\mathbf{v}_i}$ it follows that $z_i \vee x_0 \vee x_1 \equiv z_{i+1} \vee x_0 \vee x_1 \pmod{\mathbf{u}_0 \vee \mathbf{u}_1}$ $(\forall i < n)$.

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So $\mathbf{u}_j \in \operatorname{Con}_{\mathrm{c}}^{\{x_j\} \vee Z} L$ and $\mathbf{u}_j \subseteq \mathbf{a}_j \cap \operatorname{con}(x_j, 1)$. Finally, from $z_i \vee x_{\varepsilon(i)} \equiv z_{i+1} \vee x_{\varepsilon(i)} \pmod{\mathbf{v}_i}$ it follows that $z_i \vee x_0 \vee x_1 \equiv z_{i+1} \vee x_0 \vee x_1 \pmod{\mathbf{u}_0 \vee \mathbf{u}_1}$ $(\forall i < n)$. Therefore, $x_0 \vee x_1 \equiv 1 \pmod{\mathbf{u}_0 \vee \mathbf{u}_1}$.

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We shall in fact prove the following result.

Theorem

Let Ω be a set of cardinality at least $\aleph_{\omega+1}$, let L be an algebra possessing a congruence-compatible structure of a $(\vee, 1)$-semilattice, and let $\mu\colon \operatorname{Con_c} L \to \mathcal{G}(\Omega)$ be a $(\vee, 0)$-homomorphism. If μ is weakly distributive, then $\mu = 0$.

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By Růžička's work, the bound $\aleph_{\omega+1}$ can be replaced by $\aleph_2$ in the theorem above.

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At the beginning of the proof, we make the only visible use of the equalities $\mathbf{a}_0^\xi \vee \mathbf{a}_1^\xi = 1$ (for $\xi \in \Omega$).

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$$z_0^\xi = t, \quad z_{n_\xi}^\xi = 1,$$

$$\mu \operatorname{con}(z_i^\xi, z_{i+1}^\xi) \leq \mathbf{a}_{\varepsilon(i)}^\xi, \quad \text{for all } i < n_\xi.$$

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As $\aleph_{\omega+1}$ is a regular cardinal, there exists $\Omega' \subseteq \Omega$ of cardinality $\aleph_{\omega+1}$ such that $n_\xi = \text{constant}(=: n)$, for all $\xi \in \Omega'$.

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As $\aleph_{\omega+1}$ is a regular cardinal, there exists $\Omega' \subseteq \Omega$ of cardinality $\aleph_{\omega+1}$ such that $n_\xi = \text{constant}(=: n)$, for all $\xi \in \Omega'$. Pick any retraction $\rho\colon \Omega \twoheadrightarrow \Omega'$ and replace μ by $\mathcal{G}(\rho) \circ \mu$.

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As $\aleph_{\omega+1}$ is a regular cardinal, there exists $\Omega' \subseteq \Omega$ of cardinality $\aleph_{\omega+1}$ such that $n_\xi = \text{constant}(=: n)$, for all $\xi \in \Omega'$. Pick any retraction $\rho \colon \Omega \twoheadrightarrow \Omega'$ and replace μ by $\mathcal{G}(\rho) \circ \mu$. Further, we may assume that $\mu^{-1}\{0\} = \{0\}$ (replace L by its quotient by the congruence "$\mu \operatorname{con}(x, y) = 0$").

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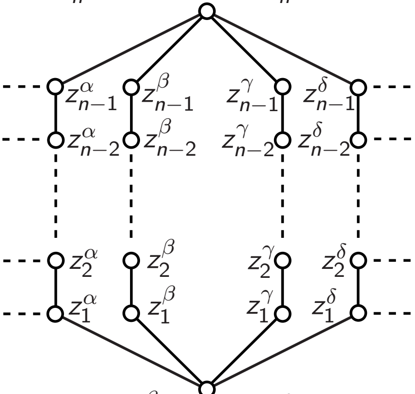
$$z_n^\alpha = z_n^\beta = z_n^\gamma = z_n^\delta = \cdots = 1$$



$$z_0^\alpha = z_0^\beta = z_0^\gamma = z_0^\delta = \cdots = t$$

$$z_i^\xi \equiv z_{i+1}^\xi \pmod{\mathbf{a}_{\varepsilon(i)}^\xi}$$

$$(\varepsilon(i) = i \mod 2)$$

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So we have reduced the problem to the case where μ separates zero and

$$z_0^\xi = t, \quad z_n^\xi = 1,$$
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We denote by $S(X)$ the join-subsemilattice of L generated by

$$\{z_i^\xi \mid 0 \leqslant i \leqslant n,\ \xi \in X\},$$

for any finite $X \subseteq \Omega$.

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$$\Phi(X) = \bigcup(\mathrm{supp}\,\mu\,\mathrm{con}(x,y) \mid x, y \in S(X))$$

is a finite subset of Ω.

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By Kuratowski's Free Set Theorem, there exists a $(2^n + 1)$-element subset H of Ω which is free with respect to Φ

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is a finite subset of Ω.

By Kuratowski's Free Set Theorem, there exists a $(2^n + 1)$-element subset H of Ω which is free with respect to Φ (more precisely, to the restriction of Φ to all 2^n-elements subsets of Ω, no change there as Φ is isotone).

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Now we pick distinct $\alpha, \beta, \delta \in H$ and apply the Erosion Lemma with z_{n-1}^{α} instead of x_0, z_{n-1}^{β} instead of x_1, and z_i^{δ} instead of z_i, for $0 \leqslant i \leqslant n$.

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Now we pick distinct $\alpha, \beta, \delta \in H$ and apply the Erosion Lemma with z^{α}_{n-1} instead of x_0, z^{β}_{n-1} instead of x_1, and z^{δ}_i instead of z_i, for $0 \leqslant i \leqslant n$.
We obtain congruences $\mathbf{u}_0 \in \mathrm{Con}_{\mathrm{c}}^{S(\{\alpha,\delta\})} L$, $\mathbf{u}_1 \in \mathrm{Con}_{\mathrm{c}}^{S(\{\beta,\delta\})} L$ such that

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Now we pick distinct $\alpha, \beta, \delta \in H$ and apply the Erosion Lemma with z_{n-1}^{α} instead of x_0, z_{n-1}^{β} instead of x_1, and z_i^{δ} instead of z_i, for $0 \leqslant i \leqslant n$.
We obtain congruences $\mathbf{u}_0 \in \mathrm{Con}_{\mathrm{c}}^{S(\{\alpha,\delta\})} L$, $\mathbf{u}_1 \in \mathrm{Con}_{\mathrm{c}}^{S(\{\beta,\delta\})} L$ such that

$$\mu(\mathbf{u}_0) \leq \mathbf{a}_0^{\delta}, \mathbf{a}_k^{\alpha},$$
$$\mu(\mathbf{u}_1) \leq \mathbf{a}_1^{\delta}, \mathbf{a}_k^{\beta},$$

with $k = \varepsilon(n-1)$, such that $\mathrm{con}(z_{n-1}^{\alpha} \vee z_{n-1}^{\beta}, 1) \leq \mathbf{u}_0 \vee \mathbf{u}_1$.

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Now we pick distinct $\alpha, \beta, \delta \in H$ and apply the Erosion Lemma with z_{n-1}^{α} instead of x_0, z_{n-1}^{β} instead of x_1, and z_i^{δ} instead of z_i, for $0 \leqslant i \leqslant n$.
We obtain congruences $\mathbf{u}_0 \in \mathrm{Con}_{\mathrm{c}}^{S(\{\alpha,\delta\})} L$, $\mathbf{u}_1 \in \mathrm{Con}_{\mathrm{c}}^{S(\{\beta,\delta\})} L$ such that

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$$\mu(\mathbf{u}_1) \leq \mathbf{a}_1^{\delta}, \mathbf{a}_k^{\beta},$$

with $k = \varepsilon(n-1)$, such that $\mathrm{con}(z_{n-1}^{\alpha} \vee z_{n-1}^{\beta}, 1) \leq \mathbf{u}_0 \vee \mathbf{u}_1$. By the definition of the set mapping Φ, we obtain

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We obtain congruences $\mathbf{u}_0 \in \mathrm{Con}_{\mathrm{c}}^{S(\{\alpha,\delta\})} L$, $\mathbf{u}_1 \in \mathrm{Con}_{\mathrm{c}}^{S(\{\beta,\delta\})} L$ such that

$$\mu(\mathbf{u}_0) \leq \mathbf{a}_0^{\delta}, \mathbf{a}_k^{\alpha},$$
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with $k = \varepsilon(n-1)$, such that $\mathrm{con}(z_{n-1}^{\alpha} \vee z_{n-1}^{\beta}, 1) \leq \mathbf{u}_0 \vee \mathbf{u}_1$. By the definition of the set mapping Φ, we obtain

$$\mu(\mathbf{u}_0) \in \mathcal{G}\Phi(\{\alpha,\delta\}),$$
$$\mu(\mathbf{u}_1) \in \mathcal{G}\Phi(\{\beta,\delta\}),$$
$$\mu\,\mathrm{con}(z_{n-1}^{\alpha} \vee z_{n-1}^{\beta}, 1) \in \mathcal{G}\Phi(\{\alpha,\beta\}).$$

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Furthermore, by the freeness of H with respect to Φ, we get that $\Phi(\{\alpha, \delta\}) \subseteq \Omega \setminus \{\beta\}$, $\Phi(\{\beta, \delta\}) \subseteq \Omega \setminus \{\alpha\}$, and $\Phi(\{\alpha, \beta\}) \subseteq \Omega \setminus \{\delta\}$.

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Furthermore, by the freeness of H with respect to Φ, we get that $\Phi(\{\alpha,\delta\}) \subseteq \Omega \setminus \{\beta\}$, $\Phi(\{\beta,\delta\}) \subseteq \Omega \setminus \{\alpha\}$, and $\Phi(\{\alpha,\beta\}) \subseteq \Omega \setminus \{\delta\}$. Therefore, by the Evaporation Lemma, we obtain that $\mu\,\mathrm{con}(z_{n-1}^{\alpha} \vee z_{n-1}^{\beta}, 1) = 0$, so, as μ separates zero, $z_{n-1}^{\alpha} \vee z_{n-1}^{\beta} = 1$.

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Furthermore, by the freeness of H with respect to Φ, we get that $\Phi(\{\alpha,\delta\}) \subseteq \Omega \setminus \{\beta\}$, $\Phi(\{\beta,\delta\}) \subseteq \Omega \setminus \{\alpha\}$, and $\Phi(\{\alpha,\beta\}) \subseteq \Omega \setminus \{\delta\}$. Therefore, by the Evaporation Lemma, we obtain that $\mu\,\mathrm{con}(z^{\alpha}_{n-1} \vee z^{\beta}_{n-1}, 1) = 0$, so, as μ separates zero, $z^{\alpha}_{n-1} \vee z^{\beta}_{n-1} = 1$.

Proceeding similarly, we obtain that $z^{\alpha}_{n-1} \vee z^{\beta}_{n-2} \vee z^{\gamma}_{n-2} = 1$, for all distinct $\alpha, \beta, \gamma \in H$.

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Proceeding similarly, we obtain that $z^{\alpha}_{n-1} \vee z^{\beta}_{n-2} \vee z^{\gamma}_{n-2} = 1$, for all distinct $\alpha, \beta, \gamma \in H$.

And then, that $z^{\alpha_0}_{n-2} \vee z^{\alpha_1}_{n-2} \vee z^{\alpha_2}_{n-2} \vee z^{\alpha_3}_{n-2} = 1$, for all distinct $\alpha_0, \alpha_1, \alpha_2, \alpha_3 \in H$. And so on. . .

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Proceeding similarly, we obtain that $z_{n-1}^{\alpha} \vee z_{n-2}^{\beta} \vee z_{n-2}^{\gamma} = 1$, for all distinct $\alpha, \beta, \gamma \in H$.

And then, that $z_{n-2}^{\alpha_0} \vee z_{n-2}^{\alpha_1} \vee z_{n-2}^{\alpha_2} \vee z_{n-2}^{\alpha_3} = 1$, for all distinct $\alpha_0, \alpha_1, \alpha_2, \alpha_3 \in H$. And so on...

At the end of the descent, we obtain that $\bigvee_{\alpha \in Y} z_0^{\alpha} = 1$, for any $Y \subseteq H$ of cardinality 2^n.

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Proceeding similarly, we obtain that $z^{\alpha}_{n-1} \vee z^{\beta}_{n-2} \vee z^{\gamma}_{n-2} = 1$, for all distinct $\alpha, \beta, \gamma \in H$.

And then, that $z^{\alpha_0}_{n-2} \vee z^{\alpha_1}_{n-2} \vee z^{\alpha_2}_{n-2} \vee z^{\alpha_3}_{n-2} = 1$, for all distinct $\alpha_0, \alpha_1, \alpha_2, \alpha_3 \in H$. And so on. . .

At the end of the descent, we obtain that $\bigvee_{\alpha \in Y} z^{\alpha}_{0} = 1$, for any $Y \subseteq H$ of cardinality 2^n. As all $z^{\alpha}_{0} = t$ (for $\alpha \in Y$), this means that $t = 1$, which concludes the proof of the main technical result.

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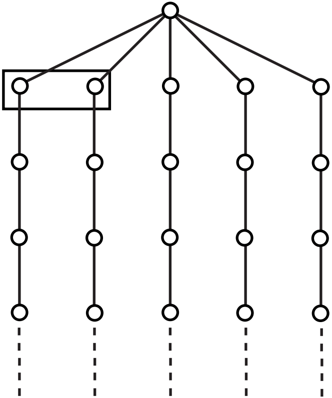

$$z_{n-1}^{\alpha_0} \vee z_{n-1}^{\alpha_1} = 1$$

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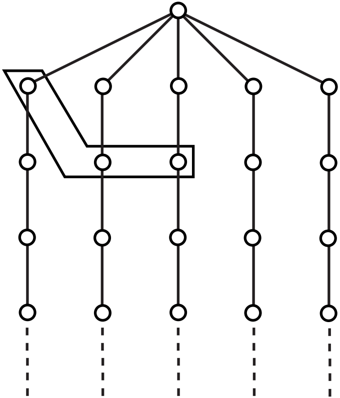

$$z_{n-1}^{\alpha_0} \vee z_{n-2}^{\alpha_1} \vee z_{n-2}^{\alpha_2} = 1$$

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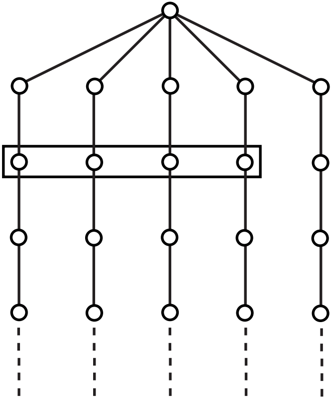

$$z_{n-2}^{\alpha_0} \vee z_{n-2}^{\alpha_1} \vee z_{n-2}^{\alpha_2} \vee z_{n-2}^{\alpha_3} = 1$$

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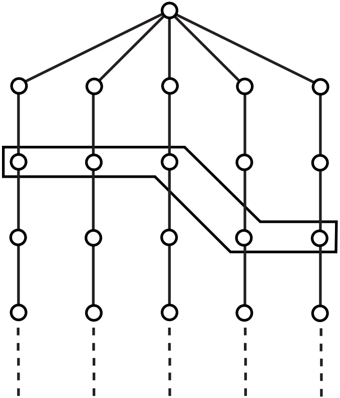

$$z_{n-2}^{\alpha_0} \vee z_{n-2}^{\alpha_1} \vee z_{n-2}^{\alpha_2} \vee z_{n-3}^{\alpha_3} \vee z_{n-3}^{\alpha_4} = 1$$

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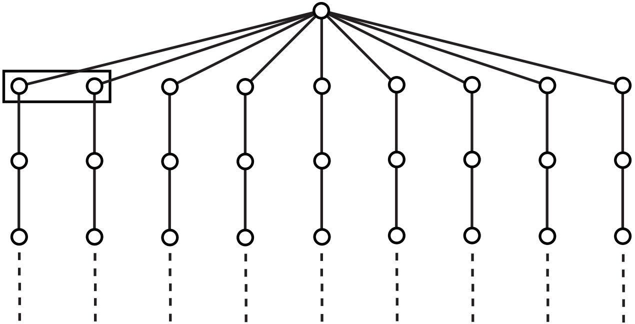

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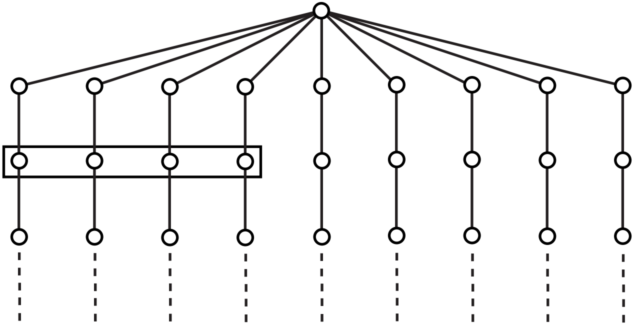

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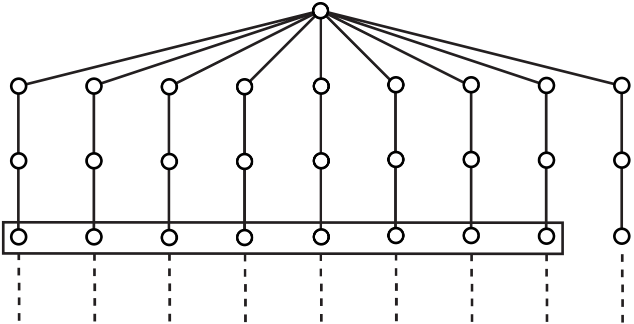

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Corollary

Let L be any algebra possessing a congruence-compatible lattice structure, let Ω be a set, $|\Omega| \geqslant \aleph_{\omega+1}$, and let $\mu\colon \operatorname{Con_c} L \to \mathcal{G}(\Omega)$ be a $(\vee, 0)$-homomorphism.

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Denote by L^{lat} the given congruence-compatible lattice structure on (the underlying set of) L. As the canonical map $\operatorname{Con_c}(L^{\mathrm{lat}}) \to \operatorname{Con_c} L$ is weakly distributive,

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Corollary

Let L be any algebra possessing a congruence-compatible lattice structure, let Ω be a set, $|\Omega| \geqslant \aleph_{\omega+1}$, and let $\mu\colon \mathrm{Con_c}\, L \to \mathcal{G}(\Omega)$ be a $(\vee, 0)$-homomorphism. If μ is weakly distributive, then $\mu = 0$.

Denote by L^{lat} the given congruence-compatible lattice structure on (the underlying set of) L. As the canonical map $\mathrm{Con_c}(L^{\mathrm{lat}}) \to \mathrm{Con_c}\, L$ is weakly distributive, it suffices to prove the corollary in case L is a lattice.

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Corollary

Let L be any algebra possessing a congruence-compatible lattice structure, let Ω be a set, $|\Omega| \geqslant \aleph_{\omega+1}$, and let $\mu\colon \operatorname{Con_c} L \to \mathcal{G}(\Omega)$ be a $(\vee, 0)$-homomorphism. If μ is weakly distributive, then $\mu = 0$.

Denote by L^{lat} the given congruence-compatible lattice structure on (the underlying set of) L. As the canonical map $\operatorname{Con_c}(L^{\mathrm{lat}}) \to \operatorname{Con_c} L$ is weakly distributive, it suffices to prove the corollary in case L is a lattice. And then, as, for all $u \leq v$ in L, the canonical map $\operatorname{Con_c}[u, v] \to \operatorname{Con_c} L$ is weakly distributive, it suffices to consider the case of bounded lattices, which holds as proved above.

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Corollary

Let L be any algebra possessing a congruence-compatible lattice structure, let Ω be a set, $|\Omega| \geqslant \aleph_{\omega+1}$, and let $\mu\colon \operatorname{Con_c} L \to \mathcal{G}(\Omega)$ be a $(\vee, 0)$-homomorphism. If μ is weakly distributive, then $\mu = 0$.

Denote by L^{lat} the given congruence-compatible lattice structure on (the underlying set of) L. As the canonical map $\operatorname{Con_c}(L^{\mathrm{lat}}) \to \operatorname{Con_c} L$ is weakly distributive, it suffices to prove the corollary in case L is a lattice. And then, as, for all $u \leq v$ in L, the canonical map $\operatorname{Con_c}[u, v] \to \operatorname{Con_c} L$ is weakly distributive, it suffices to consider the case of bounded lattices, which holds as proved above.

By Růžička's work, the assumption $|\Omega| \geqslant \aleph_{\omega+1}$ can be replaced by the assumption $|\Omega| \geqslant \aleph_2$ in the corollary above.

Getting the solution to CLP

As any isomorphism is weakly distributive, we get the announced negative solution to CLP:

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As any isomorphism is weakly distributive, we get the announced negative solution to CLP:

Corollary

If $|\Omega| \geqslant \aleph_{\omega+1}$, then there exists no lattice L such that $\mathrm{Con_c}\, L \cong \mathcal{G}(\Omega)$.

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As any isomorphism is weakly distributive, we get the announced negative solution to CLP:

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If $|\Omega| \geqslant \aleph_{\omega+1}$, then there exists no lattice L such that $\mathrm{Con_c}\, L \cong \mathcal{G}(\Omega)$.

Again, by Růžička's work, the assumption $|\Omega| \geqslant \aleph_{\omega+1}$ can be replaced by the assumption $|\Omega| \geqslant \aleph_2$.