On generic elements of mapping class groups

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Talk at the conference in honor of Patrick Dehornoy. Big thank you to Patrick !

1 Intro

Σ surface. (SAY oriented, finite genus, possibly some finite number of boundary components, possibly some finite number of punctures)
Def $\mathcal{MCG}(\Sigma)$ =group of isotopy classes of or.–pres. homeos $\Sigma \to \Sigma$. (SAY group multiplication = composition.) $\mathcal{G}$ =generating set (SAY: fixed once and for all). Cayley graph $\Gamma = \Gamma_{\mathcal{G}}(\mathcal{MCG})$
$\mathcal{CC}(\Sigma)$ = curve complex (RECALL definition)

Nielsen-Thurston classification: every $\varphi \in \mathcal{MCG}$ is

- periodic or
- reducible or
- pseudo-Anosov (pA)

φ may be periodic & reducible, no other overlap.
(SAY short explanation of each. Periodic mention Nielsen: periodic isometry of an appropriately chosen hyperbolic metric on Σ.)
(INDICATE with curly brackets:) periodic and reducible: φ acts on $\mathcal{CC}$ elliptically. pA: φ acts loxodromically.

Slogan "pseudo-Anosovs are generic": "most" elements of $\mathcal{MCG}$ are pA

Two interpretations of "most".
(SAY: long random walk model and large balls model)

(a) Long random walk in Γ (SAY: the probabilty should tend to 1)

(b) Random element from ball $B_R(1)$ in Γ, radius $R \gg 1$.

(ADD to (a):) Slogan proved, many generalisations [Igor Rivin, Joseph Maher, Alessandro Sisto] (SAY various subgroups, iwips in $Out(F_n)$)
(ADD to (b):) Slogan is a conjecture. Today: survey of what is known about this conjecture.

Weaker conjecture (Positive density of pseudo-Anosovs): $\exists R = R(\Sigma, \mathcal{G})$ s.t. any ball of radius R in Γ (SAY ball in Cayley graph, centered at *any* vertex) contains a pA element. (SAY: Corollary: the proportion of pA elements in a large ball may not tend to 1, but at least its $\liminf > 0$.)

Note Weaker conjecture true for one generating set $\mathcal{G} \implies$ true for any other g.s. $\mathcal{G}'$.

Remark added after the talk The main conjecture (that a random element from the ball $B_R(1)$ is pA with probability $\to 1$ as $R \to \infty$) does not imply the "weaker conjecture", but it implies its corollary. Thanks to Luis Paris for this clarification.

The two interpretations are different [Gouëzel, Mathéus, Maucourant][8] (SAY: They only talk about hyperbolic groups, so doesn't really apply. They prove that the two ways of picking random elements are in some sense fundamentally different.)

(SAY Weird: my feeling is that long random walks *favour (oversample)* periodic and reducible elements, so slogan should be *easier* to prove with large-balls model)

Cautionary tale Genericity of hyperbolic knots
(SAY : Thurton: knots which are not satellites and not torus knots are hyperbolic. Question: are "most" knots hyperbolic? Really depends on definition of "most"! There are at least 3 ways of defining a "random" knot. (1) Choose a random knot among all n-crossing knots. Conjecture: the proportion of hyperbolic ones tends to 1 as n tends to ∞. Has some numerical evidence for it, but a recent paper of Malyutin [9] proves it contradicts other widely believed conjecture, that crossing number is additive under connected sum. So: unknown, and at least not obviously false. (2) Constructing a random knot as a random element among all self-avoiding loops of length L in the cubic lattice $\mathbb{Z}^3$. Then "most" knots are connected sums [Soteros, Sumners, Whittington]. (3) Look at the braid group with a fixed number of strands, and take a long random braid there ("random" the large-balls sense) and take the closure, then "most" of the links obtained are hyperbolic [Caruso-Wiest].

2 The theorem of Fathi

In this section: Σ closed surface. (SAY In the direction of genericity of pseudo-Anosovs in the "large balls" model, there is one classical result:)

Theorem 1 (Fathi) *[7] If $\varphi \in \mathcal{MCG}(\Sigma)$ and if c is a simple closed curve such that the curves $\{\varphi^n(c) \mid n \in \mathbb{Z}\}$ together fill Σ, then $T_c^k \circ \varphi$ is always pseudo-Anosov, except for at most seven consecutive values of k (ARROW T_c = Dehn twist along the curve c).*

Theorem 2 (Cumplido-W) *[6] Positive density of pAs in $\mathcal{MCG}(\Sigma)$. (SAY Σ closed surface. RECALL this result works with any generating set of $\mathcal{MCG}$!)*

Lemma 3 *There is a finite set $\mathcal{C}$ of simple closed curves such that for any non-trivial, non-pseudo-Anosov element $\varphi \in \mathcal{MCG}$, there exists at least one curve $c \in \mathcal{C}$ such that c and $\varphi(c)$ together fill Σ.*

Lemma $\Longrightarrow$ Theorem (because Lemma $\overset{Fathi}{\Longrightarrow}$ for any $\varphi \in \mathcal{MCG}$, either φ is pA, or $T_c^7 \circ \varphi$ is pA, for at least one of the $c \in \mathcal{C}$).

Proof of Lemma: Use

- $a, b \in \mathcal{CC}$ fill Σ $\Leftrightarrow$ $d_{\mathcal{CC}}(a,b) \geqslant 3$
- Theorem [Bowditch [1]] $\mathcal{MCG} \curvearrowright \mathcal{CC}$ acylindrically.
- If $\varphi \in \mathcal{MCG}$, action of φ on $\mathcal{CC}$ at bounded distance from id $\Longrightarrow$ $\varphi = id$. (SAY: a proof is in [Rafi-Schleimer [10]])

TALK through the argument.

3 Using Garside theory

(SAY: all the results in this section are due to my students Sandrine Caruso, Matthieu Calvez, María Cumplido and myself, in various combinations.)

Recall Artin-Tits groups $G = A_n$ (ARROW =braid group $n+1$ strands), B_n, D_n, E_6, E_7, E_8, F_4, H_3, H_4, I_{2m} are Garside groups [Charney].

Thus every $g \in G$ has normal form $NF(g) = \Delta^k g_1 \cdot \ldots g_i \cdot g_{i+1} \cdot \ldots \cdot g_\ell$. CURLY BRACES $g_i \cdot g_{i+1}$ left-weighted (g_i Garside generators).

Def g is *rigid* if $g_\ell \cdot \tau^{-k}(g_1)$ left-weighted. (SAY: If you don't know meaning of τ, forget it. Essentially: last letter followed by first one in NF. If you know the NF of a rigid element g, the NF of g^{17} is just $NF(g)$, repeated 17 times.)

Lemma 4 *Rigid elements appear with positive density.*

SAY: Proof: rigidity is a property which depends only on the first and last letter of the NF. If you have a non-rigid element, i.e. last letter followed by first letter is not left-weighted, why is there a rigid element nearby? Because you can add a few letters to the end of the word, keeping it in normal form, so as to terminate with letter such that this last letter, followed by the first letter *is* in NF.

Construction 5 (Calvez-W) [2] of a δ-hyperbolic complex $C_{AL}(G)$ for any Garside group G on which G acts.

Moreover, if $*$ is a base point of C_{AL}, and $g = g_1 \cdot \ldots \cdot g_\ell$ in NF, then the path $* \to g_1(*) \to g_1 g_2(*) \to \ldots \to g_1 \ldots g_\ell(*)$ in C_{AL} is a quasi-geodesic. (SAY: i.e. trace of base point under a NF word is a quasi-geodesic. I mean: *unparametrized* q.geodesic, i.e. after appropriate re-parametrization, it is a quasi-geodesic. Also I mean: there is a uniform quasi-isometry constant for all these paths.)

If G Artin-Tits, then $C_{AL}(G)$ is of infinite diameter, and $\exists g \in G$ acting lox. (and w.p.d.)

If G =braid group, then

$$g \curvearrowright C_{AL} \text{ loxodromically } \overset{?}{\Longleftarrow}\!\!\!\!\!\Longrightarrow g \text{ pseudo-Anosov}$$

(SAY: Want: loxodromic actions are generic in the large-balls model)

Criterion for acting loxodr [Calvez-W] If $g = g_1 \cdot \ldots \cdot g_\ell \in G$

- is rigid
- contains a subword $g_i \cdot \ldots \cdot g_j$ $(i < j)$ s.t. $d_{C_{AL}}\big(*, g_i \ldots g_j(*)\big) > 195$

then $g \curvearrowright C_{AL}$ loxodromically. (SAY: for a *rigid* element, the guts of the action happen close to the base point. For a rigid loxodromic element, the axis passes close to the base point. A rigid element cannot act parabolically. And for a rigid elliptic element, the base point is close to the quasi-center. Thus if a rigid element wants to act elliptically, it cannot ever move the base point very far.)

Theorem 6 (Cumplido) *[5] For G an Artin-Tits group, the elements acting loxodromically have positive density.*

Also results for certain subgroups, e.g. pure braid group. (SAY: in the pure braid group, with any generating set, pAs have positive density. Also in the subgroup of braid group $\ker($*abelianisation* $B_n \to \mathbb{Z}))$

Theorem 7 (Calvez-Wiest) *[3] Large-balls-genericity of lox. acting elements in any Artin-Tits-group equipped with Garside's generating set. (SAY no idea how to do it for other generating sets. Using ideas of [Caruso-W [4]].)*

References

[1] **B. Bowditch**, Tight geodesics in the curve complex. Invent. Math. 171 (2008), no. 2, 281–300. This article proves that the $\mathcal{MCG}$-action on $\mathcal{CC}$ is acylindrical (which we need for $r=3$): $\forall r \geqslant 0$, $\exists R, N \geqslant 0$, $\forall a, b \in \mathcal{CC}$ with $d(a,b) > R$ there are at most N distinct elements φ of $\mathcal{MCG}$ such that $d(a,\varphi(a)) < r$ and $d(b,\varphi(b)) < r$.

[2] **M. Calvez, B. Wiest**, Curve complexes and Garside groups, arXiv:1503.02482, to appear in Geometriae Dedicata

[3] **M. Calvez, B. Wiest**, Acylindrical hyperbolicity and Artin-Tits groups of spherical type, arXiv:1606.07778, to appear in Geometriae Dedicata

[4] **S. Caruso, B. Wiest**, On the genericity of pseudo-Anosov braids II: conjugations to rigid braids, arXiv:1309.6137, to appear in J. Groups, Geometry, and Dynamics.

[5] **M. Cumplido**, in preparation

[6] **M. Cumplido, B. Wiest**, A positive proportion of elements of mapping class groups is pseudo-Anosov, arXiv:1703.05044

[7] **A. Fathi**, Dehn twists and pseudo-Anosov diffeomorphisms. Invent. Math., 87(1):129–151, 1987.

[8] **S. Gouëzel, F. Mathéus, F. Maucourant**, Entropy and drift in word hyperbolic groups, arXiv:1501.05082

[9] **A. Malyutin**, On the question of genericity of hyperbolic knots, arXiv:1612.03368

[10] **K. Rafi, S. Schleimer**, Curve complexes are rigid. Duke Math. J. 158 (2011), no. 2, 225–246. This article shows in particular that the only element of $\mathcal{MCG}$ which acts on $\mathcal{CC}$ such that all points are moved by a bounded amount is the identity element - but that's the easy direction, which is probably not original?