A functorial ordering of free products of ordered groups

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Ordered groups

An ordered group $(G, <)$ is a group G together with a strict total ordering $<$ of its elements such that for all $f, g, h \in G$ one has

$$f < g \iff fh < gh \iff hf < hg.$$

A homomorphism $\phi : G \to H$ of ordered groups $(G, <_G)$ and $(H, <_H)$ is order-preserving (relative to the given orderings) if for all $g, g' \in G$

$$g <_G g' \implies \phi(g) <_H \phi(g').$$

Note that ϕ is necessarily injective.
The collection of ordered groups and order-preserving homomorphisms forms a category $\mathfrak{C}$.

Free products

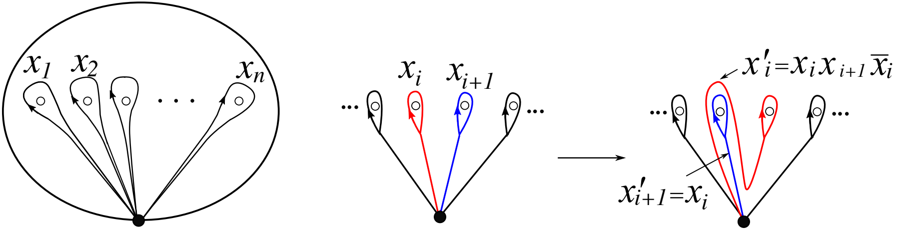
A. A. Vinogradov proved, in 1949, that if G and H are orderable groups, then the free product $G * H$ is also orderable.
Since then, several alternative proofs have appeared in the literature, including at least two incorrect proofs.
Our goal today is to prove a strong form of Vinogradov's theorem.
If $(F_i, <_{F_i}), i = 0, 1$, are ordered groups, we will construct an ordering $\prec$ of $F_0 * F_1$, so that $(F_0 * F_1, \prec)$ is an ordered group, and write

$$\mathfrak{F}((F_0, <_{F_0}), (F_1, <_{F_1})) := (F_0 * F_1, \prec).$$

Our main result is that $\mathfrak{F}$ is a functor in the category $\mathfrak{C}$.

Motivation: braid groups

The braid group B_n acts on the fundamental group of the punctured disk, which is a free group F_n.



(1) (2)

The Artin action of σ_i on $F_n \cong \langle x_1, \ldots, x_n \rangle$ is

$$x_i \to x_i x_{i+1} x_i^{-1} \quad x_{i+1} \to x_i \quad x_j \to x_j,\ j \neq i, i+1$$

Motivation: braid groups

Free groups are orderable, in uncountably many different ways, and we say that $\beta \in B_n$ is order-preserving if there exists an ordering $<$ of F_n for which the action of β is order-preserving.

One can show that

- The generators σ_i are NOT order-preserrving,
- pure braids ARE order-preserving,
- for $n > 2$ the order-preserving braids do NOT form a subgroup,
- for $n > 2$ the order-preserving braids generate B_n.

Motivation: braid groups

Tensor product of braids.

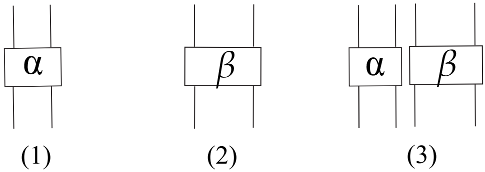


Figure: (1) $\alpha \in B_m$. (2) $\beta \in B_n$. (3) $\alpha \otimes \beta \in B_{m+n}$.

Theorem

The braid $\alpha \otimes \beta$ is order-preserving if and only if both α and β are order-preserving.

To prove this we observe that the action of $\alpha \otimes \beta$ on $F_{m+n} \cong F_m * F_n$ is a free product. That is, $\alpha \otimes \beta$ acts on F_{m+n} as

$$\alpha * \beta : F_m * F_n \to F_m * F_n.$$

If α and β preserve orderings of F_m and F_n, respectively, we need to find an ordering of $F_m * F_n$ preserved by $\alpha * \beta$.

Free products

Theorem ($\mathfrak{F}$ is a functor)

Suppose that $(F_i, <_{F_i}), i = 0, 1,$ *are ordered groups. Then the ordered group* $(F_0 * F_1, \prec_F) = \mathfrak{F}((F_0, <_{F_0}), (F_1, <_{F_1}))$ *has the following properties:*
(1) $\prec_F$ *extends the given orderings of* F_i *as subgroups of* $F_0 * F_1$*, in other words the natural inclusions are order-preserving, and*
(2) if $(G_i, <_{G_i}), i = 0, 1$ *are also ordered groups and* $(G_0 * G_1, \prec_G) = \mathfrak{F}((G_0, <_{G_0}), (G_1, <_{G_1}))$ *and if* $\phi_i : F_i \to G_i, i = 0, 1,$ *are homomorphisms which preserve the given orderings of* F_i *and* G_i*, then the homomorphism* $\phi_0 * \phi_1 : F_0 * F_1 \to G_0 * G_1$ *is order-preserving, relative to* $\prec_F, \prec_G$.

Direct products

If $(G, <_G)$ and $(H, <_H)$ are ordered groups, one can easily order the direct product $G \times H$, for example by the lexicographic ordering:

$$(g, h) <_{lex} (g', h') \iff g <_G g' \text{ or } g = g' \text{ and } h <_H h'.$$

With this ordering, the natural inclusions $G \to G \times H$ and $H \to G \times H$ are order-preserving.
Moreover, if $(G', <_{G'})$ and $(H', <_{H'})$ are ordered groups and $\phi : G \to G'$ and $\psi : H \to H'$ are order-preserving, then

$$\phi \times \psi : G \times H \to G' \times H'$$

is order preserving, relative to the lexicographic orderings.

Embedding free products in matrix rings

For free products, the situation is more difficult.
We will use a trick due to G. Bergman to construct our ordering.
Consider a ring R without zero divisors and let F and G be multiplicative groups of nonzero elements of R. Let $M_2(R[t])$ be the ring of 2×2 matrices with entries in the polynomial ring $R[t]$. Then one can embed F in $M_2(R[t])$ by $f \mapsto \left(\begin{smallmatrix} f & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

Embedding free products in matrix rings

But we can conjugate that by $\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right)$ to get a different embedding which has a highest degree in the upper right corner when $f \neq 1$:

$$\rho(f) = \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} f & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} f & (f-1)t \\ 0 & 1 \end{pmatrix}.$$

Similarly we embed G by

$$\rho(g) = \begin{pmatrix} 1 & 0 \\ (g-1)t & g \end{pmatrix}.$$

This then defines a multiplicative homomorphism $\rho : F * G \to M_2(R[t])$

Theorem (Bergman)

$\rho : F * G \to M_2(R[t])$ is injective.

Embedding free products in matrix rings

Here is a sketch of a proof using a ping-pong argument. Let $f_k g_k f_{k-1} \cdots g_2 f_1 g_1 \neq 1$ be a reduced word in $F * G$, with $f_i \in F, g_i \in G$ nonidentity elements (except possibly the first and/or last). Assume that $g_1 \neq 1$, the other case with $g_1 = 1, f_1 \neq 1$ being similar.We need to show that the product of matrices $\rho(f_k)\rho(g_k) \cdots \rho(f_1)\rho(g_1)$ is not the identity matrix.Consider the set V of column vectors $\begin{pmatrix} A(t) \\ B(t) \end{pmatrix}$ with entries in $R[t]$ and partition that set into three parts $V = V_0 \sqcup V_1 \sqcup V_2$ according to their degrees as polynomials. $V_1 : degA(t) > degB(t)$
$V_2 : degA(t) < degB(t)$ and V_0 the set with equal degrees.

Embedding free products in matrix rings

Apply $\rho(f_k)\rho(g_k)\cdots\rho(f_1)\rho(g_1)$ (on the left) to the vector $\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right) \in V_0$ and note that $\rho(g_1)$ sends $\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$ to $\left(\begin{smallmatrix}1\\g_1+(g_1-1)t\end{smallmatrix}\right)$ which belongs to V_2.Then $\rho(f_1)$ sends this result into V_1,which is then sent to V_2 by $\rho(g_2)$, and so on. The end result, after multiplying all the matrices, will be in V_1 or V_2, not V_0, and so the product cannot be the identity matrix. $\square$

Constructing the ordering of $F_0 * F_1$

Suppose we are given two ordered groups, $(F_0, <_{F_0})$ and $(F_1, <_{F_1})$. To embed them in a ring, we take R to be the integral group ring of their direct product: $R = \mathbb{Z}(F_0 \times F_1)$. It is well-known that integral group rings of orderable groups have no zero divisors, so R has no zero divisors..
Define a multiplicative homomorphism $\rho : F_0 * F_1 \to M_2(R[t])$ by

$$\rho(f_0) = \begin{pmatrix} f_0 & (f_0 - 1)t \\ 0 & 1 \end{pmatrix} \quad \rho(f_1) = \begin{pmatrix} 1 & 0 \\ (f_1 - 1)t & f_1 \end{pmatrix}, \quad f_i \in F_i.$$

Constructing the ordering of $F_0 * F_1$

First we order $F_0 \times F_1$ lexicographically; call this ordering $<$. Then the group ring $R = \mathbb{Z}(F_0 \times F_1)$ becomes an ordered ring by declaring a nonzero element to be positive if the coefficient of the largest term (in the ordering $<$ of $F_0 \times F_1$) is a positive integer.

We choose an order of the elements of a 2×2 matrix, choosing the 1,1 entry first, the 2,2 entry next and the other two entries in some fixed order. Call an element M of $M_2(R[t])$ positive if satisfies the following. Expand $M = M_0 + M_1 t + \cdots + M_k t^k$, where each M_i belongs to $M_2(R)$. Let $n \geq 0$ be the least integer such that t^n has nonzero coefficient and say M is positive iff the first nonzero entry of M_n is positive in the ordered ring R.

Finally, define the ordering $\prec$ of $F_0 * F_1$ by declaring that $x \prec y$ if and only if $\rho(y) - \rho(x)$ is positive in $M_2(R[t])$.

Constructing the ordering of $F_0 * F_1$

To check that $(F_0 * F_1, \prec)$ is an ordered group, it is easy to check that $\prec$ is a strict total ordering of the elements of $F_0 * F_1$.
Note that the diagonal elements of the 'constant' term of $\rho(F_0 * F_1)$ are always positive elements of the ring $R = \mathbb{Z}(F_0 \times F_1)$. In fact those entries are the coordinates of the image under the canonical mapping $F_0 * F_1 \to F_0 \times F_1$.
The product of such a matrix, on either side, with a positive matrix in $M_2(R[t])$ will again be positive.
Thus, if $x, y, z \in F_0 * F_1$, one has $x \prec y \iff \rho(y) - \rho(x)$ is positive $\iff (\rho(y) - \rho(x))\rho(z) = \rho(yz) - \rho(xz)$ is positive $\iff xz \prec yz$.
Left invariance is proved similarly.

Constructing the ordering of $F_0 * F_1$

We now check that $\prec$ extends the orderings of the factors.
Suppose $f_0, f_0' \in F_0$ and $f_0 <_{F_0} f_0'$. Then their images in $M_2(R[t])$ have difference the matrix $\begin{pmatrix} f_0'-f_0 & * \\ 0 & 0 \end{pmatrix}$, and noting that $f_0' - f_0$ is positive in R we conclude $f_0 \prec f_0'$. A similar argument shows that $\prec$ also extends $<_{F_1}$.

Functoriality

Recall the notation $\mathfrak{F}((F_0, <_{F_0}), (F_1, <_{F_1})) := (F_0 * F_1, \prec)$.
Rename $\prec$ as $\prec_F$.
Now suppose there are ordered groups $(G_0, <_{G_0})$ and $(G_1, <_{G_1})$ and order-preserving homomorphisms $\phi_i : F_i \to G_i$.
Consider $(G_0 * G_1, \prec_G) = \mathfrak{F}((G_0, <_{G_0}), (G_1, <_{G_1}))$.
We need to show that $\phi_0 * \phi_1 : F_0 * F_1 \to G_0 * G_1$ is order-preserving relative to the orderings $\prec_F$ and $\prec_G$.

Functoriality

Note that $\phi_0 \times \phi_1$ preserves the lexicographic orderings $<_F, <_G$ of $F_0 \times F_1$ and $G_0 \times G_1$, respectively. A homomorphism of groups naturally extends to a ring homomorphism of the integral group rings, and we see that if the group homomorphism preserves given orderings of the groups, then its extension takes "positive" elements of the group ring to positive elements.Then $\phi_0 \times \phi_1$ defines a ring homomorphism $R_F \to R_G$, where $R_F = \mathbb{Z}(F_0 \times F_1)$ and $R_G = \mathbb{Z}(G_0 \times G_1)$, which we will call $\phi_0 \times \phi_1$ again. Finally, this extends to a ring homomorphism $R_F[t] \to R_G[t]$, and further induces an additive homomorphism $M_2(R_F[t]) \to M_2(R_G[t])$, which we will again call $\phi_0 \times \phi_1$.

Functoriality

The diagram

$$\begin{array}{ccc} F_0 * F_1 & \xrightarrow{\rho} & M_2(R_F[t]) \\ {\scriptstyle \phi_0 * \phi_1}\downarrow & & {\scriptstyle \phi_0 \times \phi_1}\downarrow \\ G_0 * G_1 & \xrightarrow{\rho} & M_2(R_G[t]) \end{array}$$

is commutative (we have used the same symbol ρ for different maps, but defined analogously), and as already mentioned, $\phi_0 \times \phi_1$ takes positive matrix entries to positive matrix entries. We now argue that $\phi_0 * \phi_1$ is order-preserving, relative to $\prec_F, \prec_G$. Suppose $x, y \in F_0 * F_1$ and $x \prec_F y$.Then $\rho(y) - \rho(x)$ is positive, and therefore $\phi_0 \times \phi_1(\rho(y) - \rho(x))$ is positive in $M_2(R_G[t])$. But $\phi_0 \times \phi_1(\rho(y) - \rho(x)) =$ $\phi_0 \times \phi_1(\rho(y)) - \phi_0 \times \phi_1(\rho(x)) = \rho(\phi_0 * \phi_1(y)) - \rho(\phi_0 * \phi_1(x))$, and since this is positive, we conclude that $\phi_0 * \phi_1(x) \prec_G \phi_0 * \phi_1(y)$.

Continuity

The set of orderings $O(G)$ of the group G is endowed with a natural topology, as detailed by Sikora (2004). Consider a specific ordering $<_G$ of G, and choose a *finite* number of inequalities among elements of G which are satisfied using $<_G$. Then a basic neighbourhood of $<_G$ consists of all orderings of G for which all those inequalities remain true. Neighbourhoods of this type form a basis for the topology we are considering. Equivalently, a neighbourhood of $<_G$ is defined by choosing some finite set of elements of G which are positive (greater than the identity) using $<_G$. Then take the neighbourhood to consist of all orderings of G under which that finite set remains positive.

Continuity

If F and G are groups, we may abuse notation and regard our functor $\mathfrak{F}$ to be a mapping

$$\mathfrak{F} : O(F) \times O(G) \to O(F * G)$$

Theorem

$\mathfrak{F}$ is continuous and injective.

The crucial observation here is that to determine whether a matrix element is positive involves checking the sign of the largest element of $\mathbb{Z}(F \times G)$ in the first nonzero entry of the corresponding matrix. To be the largest element involves checking a finite number of inequalities satisfied by elements of F and G.

Arbitrary free products

By the use of (possibly transfinite) induction, our result for the free product of two ordered groups extends to the free product of an arbitrary collection of groups.

I'll conclude with a question which I think is open. Define an "order-homomorphism" $\phi : F \to G$ of ordered groups $(F, <_F)$ and $(G, <_G)$ to satisfy $f \leq_F f' \implies \phi(f) \leq_G \phi(g)$. Then enlarge our category $\mathfrak{C}$ to consist of ordered groups and order-homomorphisms.

Question: Is there an analogue of our main theorem in this larger category?

Thanks for your attention!
Thanks to the organizers!!
Happy birthday Patrick!!!