Linear representations of Artin groups and automorphisms

Luis Paris

Université de Bourgogne
Dijon, France

Journées d'algèbre 2017, En l'honneur de Patrick Dehornoy

Definition. The *braid group* on n strands, denoted by $\mathcal{B}_n$, is defined by the following presentation.

$$\mathcal{B}_n = \left\langle \sigma_1, \dots, \sigma_{n-1} \;\middle|\; \begin{array}{ll} \sigma_i\sigma_j = \sigma_j\sigma_i & \text{for } |i-j| \geq 2 \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & \text{for } |i-j| = 1 \end{array} \right\rangle .$$

One of the historical questions on the subject was whether a braid group is linear, that is, whether it admits a faithful linear representation (on $\mathbb{C}$). This question was solved by Bigelow and Krammer in the early 2000s with the following representation.

Let $\mathbb{K} = \mathbb{Q}(z, q)$, and let E be a $\mathbb{K}$-vector space of dimension $\frac{n(n-1)}{2}$ with a basis $\{e_{i,j}\}_{1 \le i < j \le n}$. For $k \in \{1, \dots, n-1\}$ we define the linear map $\psi_k : E \to E$ by

$$\psi_k(e_{i,j}) = \begin{cases} zq^2\, e_{k,k+1} & \text{if } i = k\,,\ j = k+1 \\ (1-q)e_{i,k} + q\, e_{i,k+1} & \text{if } i < k\,,\ j = k \\ e_{i,k} + zq^{k-i+1}(q-1)e_{k,k+1} & \text{if } i < k\,,\ j = k+1 \\ zq(q-1)e_{k,k+1} + q\, e_{k+1,j} & \text{if } i = k\,,\ j > k+1 \\ e_{k,j} + (1-q)e_{k+1,j} & \text{if } i = k+1\,,\ j > k+1 \\ e_{i,j} & \text{if } i < j < k \text{ or } k+1 < i < j \\ e_{i,j} + zq^{k-1}(q-1)e_{k,k+1} & \text{if } i < k < k+1 < j \end{cases}$$

Theorem (Lawrence [1990], Bigelow [2001], Krammer [2002])

The mapping $\sigma_k \mapsto \psi_k$, $1 \le k \le n-1$, induces a linear representation $\psi : \mathcal{B}_n \to \mathrm{GL}(E)$, and this representation is faithful.

The question that motivated our work is the following.

Question

How can one extend ψ to the other Artin groups?

Definition. Let S be a finite set. A *Coxeter matrix* over S is a square symmetric matrix $M = (m_{s,t})_{s,t \in S}$ indexed by S, with coefficients in $\mathbb{N}^* \cup \{\infty\}$, such that $m_{s,t} = 1$ if and only if $s = t$. The *Coxeter graph* of M is a labelled graph Γ defined as follows.

(a) S is the set of vertices of Γ.

(b) Two vertices $s, t \in S$ are linked by an edge if and only if $m_{s,t} \geq 3$, and this edge is labelled by $m_{s,t}$ if $m_{s,t} \geq 4$.

Definition. For two letters a, b and an integer $m \geq 2$ we denote by $\langle ab \rangle^m$ the word $aba \cdots$ of length m.

(a) The *Artin group* associated with Γ is

$$A = A_\Gamma = \langle S \mid \langle st \rangle^{m_{s,t}} = \langle ts \rangle^{m_{s,t}} \text{ for } s, t \in S,\ s \neq t,\ \text{and } m_{s,t} \neq \infty \rangle\,.$$

(b) The *Artin monoid* associated with Γ is

$$A^+ = A^+_\Gamma = \langle S \mid \langle st \rangle^{m_{s,t}} = \langle ts \rangle^{m_{s,t}} \text{ for } s, t \in S,\ s \neq t,\ \text{and } m_{s,t} \neq \infty \rangle^+\,.$$

(c) The *Coxeter group* associated with Γ, denoted by $W = W_\Gamma$, is the quotient of A by the relations $s^2 = 1$, $s \in S$.

Example. Let $\Gamma = A_{n-1}$ be the following Coxeter graph.

Then

$$A_\Gamma = \left\langle s_1, \dots, s_{n-1} \;\middle|\; \begin{array}{ll} s_i s_j = s_j s_i & \text{for } |i-j| \ge 2 \\ s_i s_j s_i = s_j s_i s_j & \text{for } |i-j| = 1 \end{array} \right\rangle .$$

This is the braid group $\mathcal{B}_n$. W_Γ is the quotient of $\mathcal{B}_n$ by the relations $s_i^2 = 1$, $1 \le i \le n-1$. It is isomorphic to the symmetric group $\mathfrak{S}_n$.

Definition. We say that Γ is

(a) of *spherical type* if W_Γ is finite,

(b) *simply laced* if $m_{s,t} \in \{2, 3\}$ for all $s, t \in S$, $s \ne t$,

(c) *triangle free* if there are no $s, t, r \in S$, pairwise distinct, such that $m_{s,t}, m_{s,r}, m_{t,r} \ge 3$.

"Theorem" (Cohen–Wales [2002], Digne [2003], Paris [2002])

The construction of the Krammer representation extends to simply laced and triangle free Artin groups.

(1) If Γ is simply laced and triangle free, then this representation $\psi : A_\Gamma^+ \to \mathrm{GL}(E)$ is faithful on the Artin monoid.

(2) If Γ is simply laced and of spherical type, then the whole representation $\psi : A_\Gamma \to \mathrm{GL}(E)$ is faithful, and E is of finite dimension.

Open question

Up to a few exceptions, it is not known whether an Artin group A_Γ is always linear.

Definition. Let $\Pi = \{\epsilon_s \mid s \in S\}$ be a set in one-to-one correspondence with S, and let $V = \oplus_{s\in S}\mathbb{R}\,\epsilon_s$ be the real vector space with basis Π. Let $\langle .,. \rangle : V \times V \to \mathbb{R}$ be the symmetric bilinear form define by

$$\langle \epsilon_s, \epsilon_t \rangle = \begin{cases} -\cos(\pi/m_{s,t}) & \text{if } m_{s,t} \neq \infty\,, \\ -1 & \text{if } m_{s,t} = \infty\,. \end{cases}$$

For all $s \in S$ we define the linear transformation $f_s : V \to V$ by

$$f_s(x) = x - 2\langle x, \epsilon_s \rangle \epsilon_s\,.$$

Then the mapping $S \to \mathrm{GL}(V)$, $s \mapsto f_s$, induces a faithful linear representation $f : W_\Gamma \to \mathrm{GL}(V)$, called the *canonical representation* of W_Γ.

Notation. From now on, for $w \in W_\Gamma$ and $x \in V$, the vector $f(w)(x)$ will be denoted by $w\,x$.

Definition. The set $\Phi = \{w\,\epsilon_s \mid w \in W_\Gamma\,,\ s \in S\}$ is called the *root system* of Γ. Let $\rho = \sum_{s \in S} \lambda_s \epsilon_s \in \Phi$ be a root. We say that ρ is a *positive root* if $\lambda_s \geq 0$ for all $s \in S$. We denote by Φ^+ the set of positive roots, and we set $\Phi^- = -\Phi^+$.

Fact

$\Phi = \Phi^+ \sqcup \Phi^-$.

Definition. Let $\mathbb{K} = \mathbb{Q}(q, z)$, and let $\mathcal{B} = \{e_\rho \mid \rho \in \Phi^+\}$ be a set in one-to-one correspondence with Φ^+. Let $E = \oplus_{\rho \in \Phi^+} \mathbb{K}\, e_\rho$ be a $\mathbb{K}$-vector space with basis $\mathcal{B}$. For $s \in S$ we define the linear map $\varphi_s : E \to E$ by

$$\varphi_s(e_\rho) = \begin{cases} 0 & \text{if } \rho = \epsilon_s \\ e_\rho & \text{if } \langle \rho, \epsilon_s \rangle = 0 \\ q\, e_{s\rho} & \text{if } \langle \rho, \epsilon_s \rangle > 0 \text{ and } \rho \neq \epsilon_s \\ (1-q)e_\rho + e_{s\rho} & \text{if } \langle \rho, \epsilon_s \rangle < 0 \end{cases}$$

Fact

(1) We have

$$\begin{aligned} \varphi_s\varphi_t &= \varphi_t\varphi_s & \text{if } m_{s,t} = 2 \\ \varphi_s\varphi_t\varphi_s &= \varphi_t\varphi_s\varphi_t & \text{if } m_{s,t} = 3 \end{aligned}$$

(2) φ_s is not invertible.

Proposition (Cohen–Wales [2002], Digne [2003], Paris [2002])

We assume that Γ is simply laced and triangle free. Then there exists a collection of polynomials $T(s,\rho) \in \mathbb{Z}[q]$, $s \in S$ and $\rho \in \Phi^+$, such that the linear map $\psi_s : E \to E$ defined by

$$\psi_s(e_\rho) = \varphi_s(e_\rho) + z\, T(s,\rho)\, e_{\epsilon_s}$$

satisfies the following properties.

(1) $\psi_s \in \mathrm{GL}(E)$, for all $s \in S$.

(2) We have

$$\begin{array}{rcll} \psi_s\psi_t & = & \psi_t\psi_s & \text{if } m_{s,t} = 2 \\ \psi_s\psi_t\psi_s & = & \psi_t\psi_s\psi_t & \text{if } m_{s,t} = 3 \end{array}$$

We denote by $\psi : A_\Gamma \to \mathrm{GL}(E)$ the linear representation which sends s to ψ_s for all $s \in S$.

Linear representation

Theorem (Cohen–Wales [2002], Digne [2003], Paris [2002])

(1) Let Γ be a simply laced Coxeter graph of spherical type. Then $\psi : A_\Gamma \to \mathrm{GL}(E)$ is faithful.

(2) Let Γ be a simply laced and triangle free Coxeter graph. Then the restriction of ψ to A_Γ^+, $\psi^+ : A_\Gamma^+ \to \mathrm{GL}(E)$, is faithful.

Questions

(1) Let Γ be a simply laced and triangle free Coxeter graph. Suppose that Γ is not of spherical type. Is $\psi : A_\Gamma \to \mathrm{GL}(E)$ faithful?

(2) Assume that Γ is not simply laced. Let $E = \oplus_{\rho \in \Phi^+} \mathbb{K}\, e_\rho$ be a $\mathbb{K}$-vector space with basis $\mathcal{B} = \{e_\rho \mid \rho \in \Phi^+\}$. Does there exist a linear representation $\psi : A_\Gamma \to \mathrm{GL}(E)$ which is faithful (at least on A_Γ^+)?

Let Γ be any Coxeter graph. Recall that, for $s \in S$, the linear map $\varphi_s : E \to E$ is defined by

$$\varphi_s(e_\rho) = \begin{cases} 0 & \text{if } \rho = \epsilon_s \\ e_\rho & \text{if } \langle \rho, \epsilon_s \rangle = 0 \\ q\, e_{s\rho} & \text{if } \langle \rho, \epsilon_s \rangle > 0 \text{ and } \rho \neq \epsilon_s \\ (1-q)e_\rho + e_{s\rho} & \text{if } \langle \rho, \epsilon_s \rangle < 0 \end{cases}$$

Proposition [Unpublished]

Let $s, t \in S$ such that $m_{s,t} \neq \infty$. Then

$$\langle \varphi_s \varphi_t \rangle^{m_{s,t}} = \langle \varphi_t \varphi_s \rangle^{m_{s,t}} .$$

Bad news

Explicit calculations indicate that there are no polynomials $T(s,\rho) \in \mathbb{Z}[q]$, $s \in S$ and $\rho \in \Phi^+$, such that the linear maps $\psi_s : E \to E$ defined by

$$\psi_s(e_\rho) = \varphi_s(e_\rho) + z\, T(s,\rho)\, e_{\epsilon_s}$$

satisfy the following properties.

(1) $\psi_s \in \mathrm{GL}(E)$, for all $s \in S$.

(2) We have

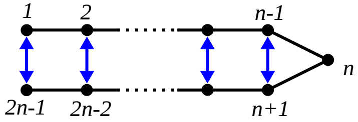
$$\langle \psi_s \psi_t \rangle^{m_{s,t}} = \langle \psi_t \psi_s \rangle^{m_{s,t}}$$

for all $s,t \in S$ such that $m_{s,t} \neq \infty$.

Other bad news

Even if the polynomials $T(s, \rho)$ existed, the proof of the faithfulness of ψ (which depends only on φ) would not extend to non simply laced Artin groups.

Good news. Let $\Gamma = A_{2n-1}$, and let g be the automorphism of Γ defined by $g(s_i) = s_{2n-i}$.



Then g acts on $V = \oplus_{s \in S} \mathbb{R}\, \epsilon_s$ and $g(\Phi^+) = \Phi^+$. So, we have an action of g on $E = \oplus_{\rho \in \Phi^+} \mathbb{K}\, e_\rho$. On the other hand, g acts on A. We set $A^g = \{\alpha \in A \mid g(\alpha) = \alpha\}$ and $E^g = \{x \in E \mid g(x) = x\}$. The representation $\psi : A \to \mathrm{GL}(E)$ is equivariant in the sense that

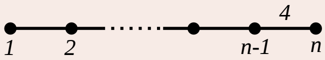
$$\psi(g\,\alpha) = g\,\psi(\alpha)\,g^{-1}$$

for all $\alpha \in A$, hence ψ induces a linear representation $\psi^g : A^g \to \mathrm{GL}(E^g)$.

Automorphism

Theorem (Michel [1999], Dehornoy–Paris [1999], Crisp [2000], Digne [2003])

(1) Let $\tilde{\Gamma} = B_n$ be the the following Coxeter graph.



Then A^g is the Artin group associated with $\tilde{\Gamma}$.

(2) Let $\tilde{\Phi}^+$ be the positive root system associated with $\tilde{\Gamma}$. Then E^g has a natural basis $\tilde{\mathcal{B}}$ in one-to-one correspondence with $\tilde{\Phi}^+$, and the linear representation $\psi^g : A^g \to \mathrm{GL}(E^g)$ is faithful.

Automorphism

Let Γ be a simply laced and triangle free Coxeter graph, and let G be a group of automorphisms of Γ. Then G acts on $V = \oplus_{s \in S} \mathbb{R}\, \epsilon_s$, and $g(\Phi^+) = \Phi^+$ for all $g \in G$. Hence, we have an action of G on $E = \oplus_{\rho \in \Phi^+} \mathbb{K}\, e_\rho$. On the other hand, G acts on A. We set $A^G = \{\alpha \in A \mid g(\alpha) = \alpha \text{ for all } g \in G\}$ and $E^G = \{x \in E \mid g(x) = x \text{ for all } g \in G\}$. The linear representation $\psi : A \to \mathrm{GL}(E)$ is equivariant, hence it induces a linear representation $\psi^G : A^G \to \mathrm{GL}(E^G)$.

Theorem (Crisp [2000], Castella [2006])

(1) Let $A^{+G} = \{\alpha \in A^+ \mid g(\alpha) = \alpha \text{ for all } g \in G\}$. Then A^{+G} is an Artin monoid.

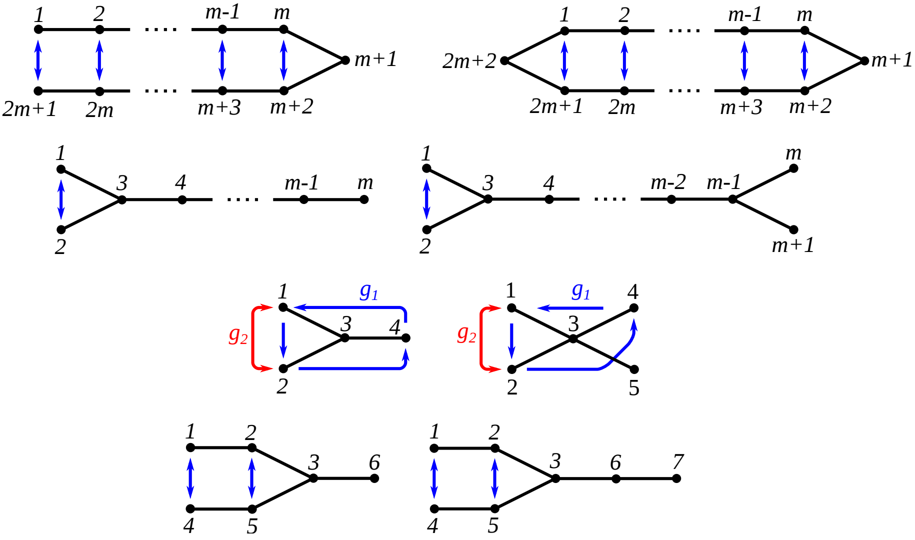
(2) The restriction of ψ^G to A^{+G}, $\psi^{+G} : A^{+G} \to \mathrm{GL}(E^G)$, is faithful.

Question

We know that A^G is an Artin group in many cases (when A is of spherical type, of FC type, son on ...), but not always.

Theorem (Geneste–Paris [$\geq$ 2017])

Let $\tilde{\Phi}^+$ be the positive root system of A^{+G}. There exists a "natural" linearly independent subset $\tilde{\mathcal{B}}$ of E^G in one-to-one correspondence with $\tilde{\Phi}^+$. The set $\tilde{\mathcal{B}}$ is a basis of E^G if and only if (Γ, G) is one of the following eight pairs, up to isomorphism.


1
2
m-1
m
m+1
2m+1
2m
m+3
m+2
2m+2
1
2
3
4
m-1
m
m-2
m-1
m+1
g1
g2
1
2
3
4
5
6
7

Thank you for your attention!