Intrinsic geometry of the dual braid complex

Jon McCammond

UC Santa Barbara

Caen
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Dual structure

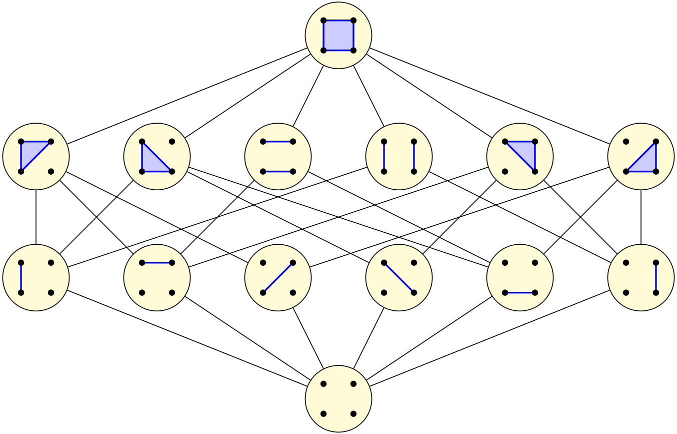
Definition (Dual Garside structure)

The dual Garside structure for the braid group $\textsc{Braid}_n$ is build around the minimum length factorizations of a fixed n-cycle into reflections. These factorizations can be identified with maximal chains in the noncrossing partition lattice NCPart_n.

Remark (Dual braid complex)

From this dual Garside structure one can build a contractible simplicial complex with a free $\textsc{Braid}_n$ action that I call the dual braid complex. It is built out of copies of the geometric realization of the noncrossing partition lattice and this finite subcomplex forms a strong fundamental domain for the action.

Noncrossing partitions

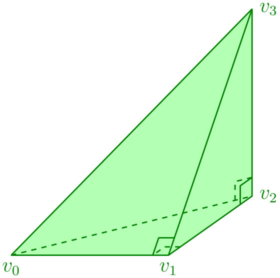


Orthoscheme

Definition (Orthoscheme)

An orthoscheme is a convex hull of a piecewise geoedesic path in $\mathbb{R}^n$ where the vectors associated to each geoedesic portion are pairwise orthogonal. When all of these vectors are unit vectors, this is a standard orthoscheme.

Here is a 3-dimensional orthoscheme.



The cross-section complex and its vertex link

Definition (Cross-section complex)

When the dual braid complex is given an orthoscheme metric, the orthoschemes form columns and the metric space splits as a direct product of the real line with a simpler complex one dimension lower that I call the cross-section complex.

Definition (Noncrossing partition link)

All vertex links in the cross-section complex are identical and equal to the link of the edge connecting the bounding vertices in the geometric realization of the noncrossing partition lattice with the orthoscheme metric. I call this piecewise spherical complex the noncrossing partition link.

Low-dimensional examples

Example ($\textsc{Braid}_3$)

For $\textsc{Braid}_3$ its dual braid complex is (metrically) $T_3 \times \mathbb{R}$ where T_3 is an infinite 3-regular tree, its cross-section complex is T_3 and its noncrossing partition link has only 3 discrete points.

Example ($\textsc{Braid}_4$)

For $\textsc{Braid}_4$ its dual braid complex is made of 3-orthoschemes, its cross-section complex is built out of equilateral triangles and its vertex link is built out of arcs of length $\frac{\pi}{3}$.

Remark ($\textsc{Braid}_{n+1}$)

For $\textsc{Braid}_{n+1}$ these spaces are made of n-orthoschemes, PE $\widetilde{A}_{n-1}$ Coxeter shapes and PS A_{n-1} Coxeter shapes.

Associahedra

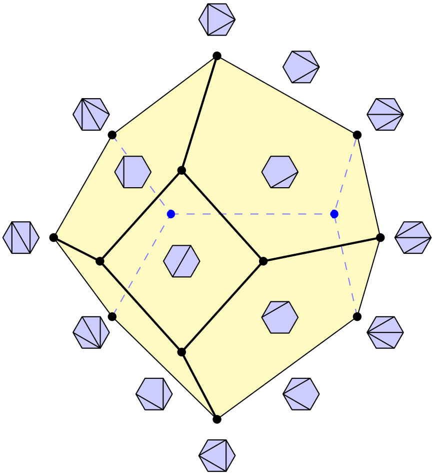
Definition (Simple Associahedron)

Let Γ be a graph whose vertices correspond to triangulations of a fixed convex polygon by adding noncrossing diagonals and whose edges correspond to triangulations that only differ in the placement of a single diagonal. The simple associahedron is the convex polytope whose 1-skeleton is Γ.

Definition (Simplicial Associahedron)

The simplicial associahedron is the polytope dual to the simple one. Its facets are labeled by the triangulations, its vertices are labeled by diagonal edges and its edges correspond to a pair of noncrossing diagonals.

The 3-dimensional simple associahedron



Connections with noncrossing partitions

Remark (Bijections)

There are various known bijections between noncrossing partitions and associahedra. Both have objects counted by the Catalan numbers and further refined partitions counted by the Narayana numbers.

Remark (Unsatisfying)

These bijections have been known for some time (since at least the early 2000s) but I have always found them slightly unsatisfying. Why is there such a close connection?

By the end of this talk I'll give an answer I find satisfying.

Generalizations I

Remark (Noncrossing partitions)

Once noncrossing partitions were connected to Garside structures for the symmetric group, it was easy to define a noncrossing partion lattice $\mathrm{NCPart}(W, c)$ for every finite Coxeter group W and choice of Coxeter element c.

Remark (Associahedra)

Once associhedra were connected to Formin and Zelevinsky's theory of cluster algebras, it was easy to define a generalized associahedron $\mathrm{Assoc}(W, c)$ for every finite Coxeter group W and choice of Coxeter element c.

The known bijections extend to all of these cases.

Generalizations II

Remark (Adding an integer parameter)

There are also general versions that adds an integer parameter m. One defines $\mathrm{NCPart}^{(m)}(W,c)$ and $\mathrm{Assoc}^{(m)}(W,c)$ for all choices of finite Coxeter group W, Coxeter element c and positive integer m. The classical version is when $m=1$. These constructions nest so that $\mathrm{Assoc}^{m}(W,c) \hookrightarrow \mathrm{Assoc}^{m+1}(W,c)$.

Example (The 2-associahedron)

The higher level associahedra are no longer polytopes. They are merely simplicial complexes. The level 2 associahedron for the symmetric group $\mathrm{Assoc}^{(2)}(\textsc{Sym}_n,\delta)$, for example, is the complex of partitions of a $2n$-gon into even-sided subpolygons.

Generalizations III

Definition (Sortable elements)

Nathan Reading introduced a collection of elements that he called sortable elements associated to any finite Coxeter group W and choice of Coxeter element c. These were extended to all Coxeter groups and there is a version with an added integer parameter m.

Theorem (Stump-Thomas-Williams)

There are natural bijections between the generalizations of noncrossing partitions, associahedra and sortable elements for all finite Coxeter groups W, all choices of Coxeter element c and all positive integers m.

Noncrossing trees

Let's return to the braid group.

Remark (Factorizations and the link)

A maximal chain in $\mathrm{NCPart}(\mathrm{S}\textsc{ym}_{n+1})$ records a factorization into transpositions. When these transpositions are draw on a convex $(n+1)$-gon whose vertices are labeled according to the chosen Coxeter element, the result is a noncrossing tree. To be able to recover the original factorization we should label the edges with numbers indicating where they occur in the factorization. We call this an ordered noncrossing tree.

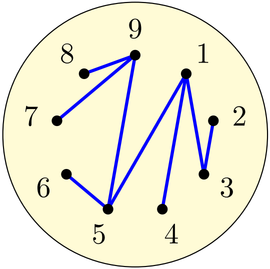
Remark

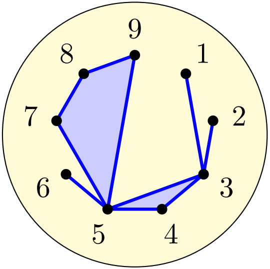
A non-maximal chain will correspond to something that I'll call a properly ordered noncrossing hypertree.

Noncrossing hypertrees

Definition (Hypertrees)

A hypergraph has a vertex set and a set of hyperedges (subsets of vertices of size at least 2). A hypertree is a connected hypergraph with no hypercycles. It is noncrossing when the hyperedge convex-hulls of intersect in at most a vertex. A noncrossing tree and noncrossing hypertree are shown.





Noncrossing hypertree complex

Definition (Noncrossing hypertree complex)

There is an ordering on the set of noncrossing hypertrees that turns it into the face poset of a simplicial complex. I call this the noncrossing hypertree complex. The f-vector and reduced euler characteristic of $\mathrm{Complex}(\mathrm{NCHT}_{n+1})$ for small values of n.

n	f_{-1}	f_0	f_1	f_2	f_3	f_4	f_5	$\widetilde{\chi}$
2	1	3						2
3	1	8	12					–5
4	1	15	55	55				14
5	1	24	156	364	273			–42
6	1	35	350	1400	2380	1428		132
7	1	48	680	4080	11628	15504	7752	–429

Proper Orderings I

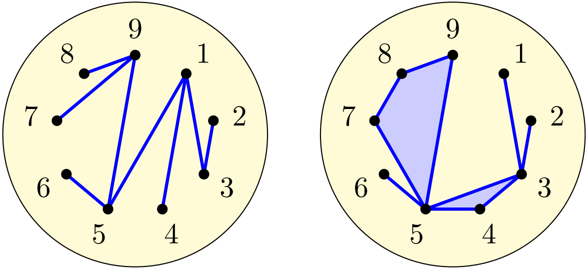
Remark (Restrictions)

The ordering on the hyperedges of a noncrossing hypertree is not arbitrary. There are necessary and sufficient restrictions that ensure their ordered product is the correct element.

Definition (Edge posets)

The hyperedges sharing a particular vertex, for example, must occur in the local linear ordering visible at that vertex. For each noncrossing hypertree τ there is an edge poset $\text{Poset}(\tau)$ that records these restrictions. The proper orderings of the hyperedges of a noncrossing hypertree correspond to the linearizations of its edge poset.

Proper Orderings II



In this noncrossing tree 56 < 59 < 15 is visible at 5 and 13 < 14 < 15 visible at 1. In this noncrossing hypertree 56 < 5789 < 345 is visible at 5 and 345 < 13 < 23 is visible at 3. In particular, the hypertree has a unique linearization.

Intrinsic geometry I

Definition (Intrinsic geometry)

Intrinsic geometry is the structure visible in the metric. For example, the intrinsic structure of a graph is found by removing degree 2 vertices and merging edges.

Remark (Finding the intrinsic geometry)

When 2 top-dimensional simplices share a common facet and they are the only ones that contain this facet, then the common facet can be erased to form a single larger cell. Continue.

Remark (Potential problems)

Imagine this process for a 2-complex. The edges might become circles and the triangles might become polygons or annuli or even more complicated surfaces.

Intrinsic geometry II

In the noncrossing partition link no problems arise.

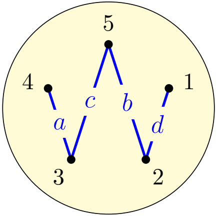
Theorem (Intrinsic geometry)

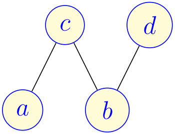
The intrinsic geometry of the noncrossing partition link is a simplicial complex with bigger simplices. In fact, two simplices labeled by properly ordered noncrossing hypertrees will be merged in the intrinsic structure if and only if they are proper orderings of the exact same noncrossing hypertree.

In short the intrinsic geometry on the noncrossing partition link is the noncrossing hypertree complex. Here are two examples in low dimensions.

A small example I

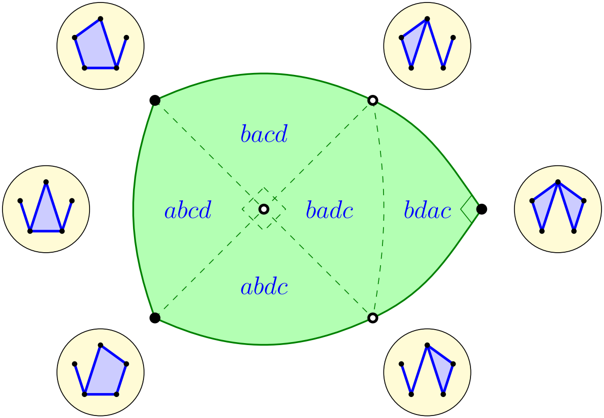
A noncrossing tree τ with 5 vertices and 4 edges is shown on the left. Its hyperedge poset Poset(τ) is shown on the right.



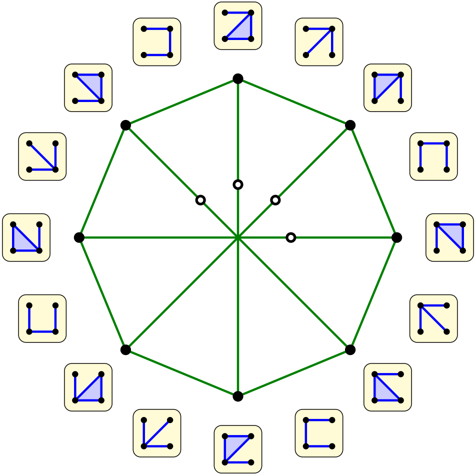


There are exactly 5 linearizations of Poset(τ) corresponding to the words *abcd*, *abdc*, *bacd*, *badc* and *bdac*. In the noncrossing partition link these label 5 spherical triangles that fit together as follows. Notice that the faces of this simplex are labeled by noncrossing hypertrees.

A small example II



Link($NCPart_4$) vs. Complex($NCHT_4$)



Comparing the 2 simplicial structures

Number of vertices and chambers for the structures in low dims.

n	d	$\|V\|$	chambers
3	1	12	16
4	2	40	125
5	3	130	$1,296$
6	4	427	$16,807$
7	5	$1,428$	$262,144$
8	6	$4,860$	$4,782,969$
9	7	$16,794$	$100,000,000$

n	d	$\|V\|$	chambers
3	1	8	12
4	2	15	55
5	3	24	273
6	4	35	$1,428$
7	5	48	$7,752$
8	6	63	$43,263$
9	7	80	$246,675$

Left: noncrossing partition link $\text{Link}(\text{NCPart}_{n+1})$
Right: noncrossing hypertree complex $\text{Complex}(\text{NCHT}_{n+1})$

Simplices and spheres

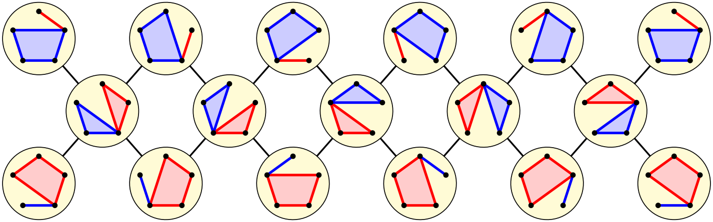
It was already known that the noncrossing partition link is a union of apartments in a spherical building and that these apartments can be bijectively labeled by noncrossing trees.

Theorem (Simplices and Spheres)

In the simplified cell structure noncrossing trees label the top-dimensional simplices called chambers as well as the top-dimensional metric unit spheres called apartments.

Vertices

The 15 basic noncrossing hypertrees on 5 vertices. Note that left and right should be glued to form a cyclinder.



The lines indicate covering relations in the noncrossing partition lattice. The number of vertices only grows quadratically.

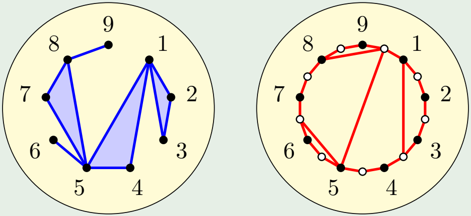
Associahedra I

Remark (Noncrossing hypertrees and level 2 associahedra)

There is a natural bijection between the noncrossing hypertree complex and the level 2 associahedron.

Example

A noncrossing hypertree on 9 vertices and the corresponding dissection of an 18-gon into even-sided subpolygons.



Associahedra II

This is the satisfactory answer I was looking for. There is strong connections between noncrossing partitions and associahedra because **the noncrossing partition link is intrinsically a level** 2 **associahedron.** This means the simplified cell structure should contain simplicial associahedral apartments and it does.

Theorem (Associahedra)

Let τ be a noncrossing tree. The apartment labeled τ is a simplicial associahedron iff $\mathrm{Poset}(\tau)$ *has a unique linearization.*

Hohlweg and Lange constructed a variety of simple metric associahedra from cluster algebras and they have simplicial associahedral normal fans. All of these occur in the intrinsic geometry of the noncrossing partition link.

A strange duality

The noncrossing partition link exhibits a kind of strange duality.

Theorem (Bijections)

For each noncrossing tree τ there is a bijection between the set of apartments containing the simplex labeled τ and the set of tree simplices contained in the apartment labeled τ.

In symbols:

$$\#\{\sigma \mid \text{Chamber}(\tau) \in \text{Apart}(\sigma)\}$$
$$= \#\{\sigma \mid \text{Chamber}(\sigma) \in \text{Apart}(\tau)\}$$

Finally, the simplifications we did to the vertex link can also be extended to the full cross-section complex.

Intrinsic geometry of the dual braid complex

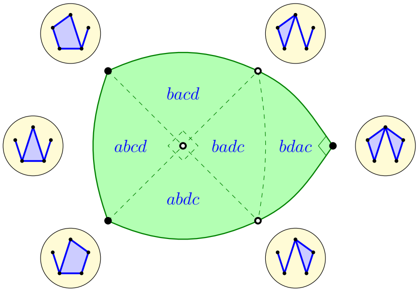
Definition (Polysimplices)

A *polysimplex* is a finite direct product of simplices. When considering *metric polysimplices* we insist that the simplices be euclidean but we allow slanted products. For example all parallelograms count are metric polysimplices that are products or two intervals.

Theorem (Polysimplices in the cross-section)

The intrinsic simplicial structure on the noncrossing partition link extends to an intrinsic metric polysimplicial structure on the cross-section complex. And the intrinsic geometry of the dual braid complex is built out of metric polysimplices cross $\mathbb{R}$.

Noncrossing hypertrees



Remark (Preprint)

These results are in the preprint “Noncrossing hypertrees”.

It is available on my webpage but it is not (yet) on the arxiv.