Gcd-monoids arising from categories and posets

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March 2017

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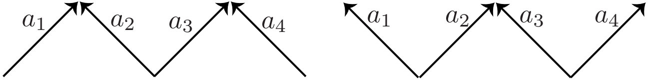

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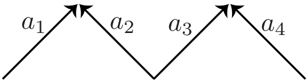

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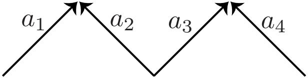

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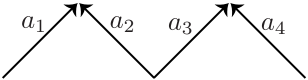

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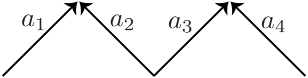

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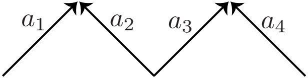

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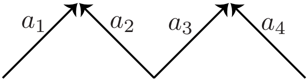

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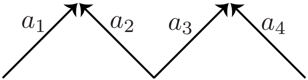

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- However, it represents 1 in the enveloping group $\tilde{A}_2 = \mathrm{U}_{\mathrm{gp}}(\tilde{A}_2^+)$.
- Hence a better concept of reduction is needed.

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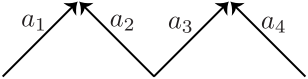

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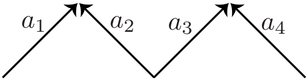

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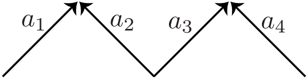

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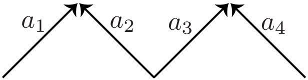

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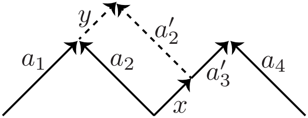

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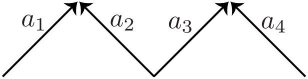

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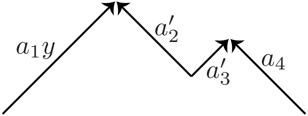

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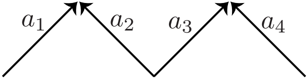

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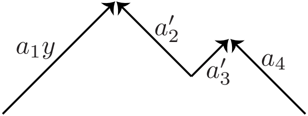

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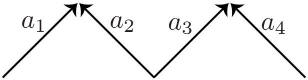

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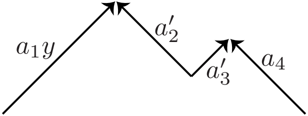

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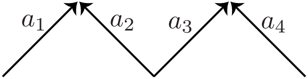

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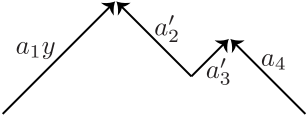

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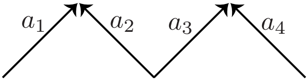

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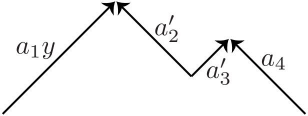

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- If the process stops, we get an irreducible multifraction.
- This reduction system may not be convergent: a multifraction may reduce to two distinct irreducible multifractions (example in $\tilde{A}_2^+$: $1/c/aba$ reduces to both $ac/ca/ba$ and $bc/cb/ab$).

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- Example (Brieskorn-Saito, 1972): Artin-Tits monoids, that is, monoids defined by sets of relations of the form $aba \cdots = bab \cdots$ (same length on both sides, one relation for each pair $\{a, b\}$).

Dehornoy's question on Artin-Tits monoids

- Say that reduction, in a gcd-monoid M, is semi-convergent if for every multifraction $\underline{a}$ of M, $\underline{a}$ reduces to 1 iff $\underline{a}$ represents 1 in $\mathrm{U}_{\mathrm{gp}}(M)$.

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- How well do things work out for general gcd-monoids?

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Proof (involves highlighting trick).

The elements $x^{-1}y \in \mathrm{F_{gp}}(P)$, where $x \leq y$ in P, satisfy the defining relations of $\Upsilon(P)$. Thus there exists a unique monoid homomorphism $\mu \colon \Upsilon(P) \to \mathrm{F_{gp}}(P)$ such that each $\mu([x, y]) = x^{-1}y$. Any $\boldsymbol{a} \in \Upsilon(P)$ can be written $[x_1, y_1] \cdots [x_n, y_n]$ with each $y_i \neq x_{i+1}$ (reduced word). Then $\mu(\boldsymbol{a}) = x_1^{-1}y_1 \cdots x_n^{-1}y_n$. The right hand side is a word in normal form in $\mathrm{F_{gp}}(P)$. $\square$

Interval monoids and groups

- The interval monoid $\Upsilon(P)$ (resp., interval group $\Upsilon^{\pm}(P)$), of a poset P, is defined by generators $[x, y]$, for $x \leq y$ in P, and relations $[x, x] = 1$, $[x, z] = [x, y] \cdot [y, z]$ for $x \leq y \leq z$.

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For posets $P \subseteq Q$, every reduced word $[x_1, y_1] \cdots [x_n, y_n]$ in $\Upsilon(P)$ (i.e., each $y_i \neq x_{i+1}$) is also a reduced word in $\Upsilon(Q)$. Hence,

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For any finite poset P, with n elements, $\Upsilon(P)$ embeds into the free monoid $\mathrm{F}_{\mathrm{mon}}(n-1)$.

When are they gcd-monoids?

For any poset P and any $a \in P$, we set

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Floating homotopy group of a simplicial complex

- A set K, of nonempty finite subsets of some set V (the vertices of K), is a simplicial complex on V, if $\{p\} \in K$ for all $p \in V$, and $(\varnothing \neq X \subseteq Y$ and $Y \in K)$ implies $X \in K$. The elements of K are called the simplices of K.

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- Hence $\Upsilon^{\pm}(K) = \mathrm{U}_{\mathrm{gp}}(\Pi_1(K))$, the universal group of the fundamental groupoid $\Pi_1(K)$ of K.

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Every group is a doubly free factor of $\Upsilon^{\pm}(K)$, for some connected simplicial complex K of dimension at most 2.

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- If K has dimension at most 2, then P is a connected poset of length at most 2.
- Moreover, it is well known that $\pi_1(K) \cong \pi_1(\mathrm{Sim}(P))$.

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- In particular, Every group is a doubly free factor of the universal group of an interval gcd-monoid.

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- In particular, There are finite posets P, of length 2, for which $\Upsilon(P)$ is a gcd-monoid where reduction is not semi-convergent.

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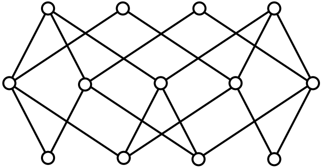

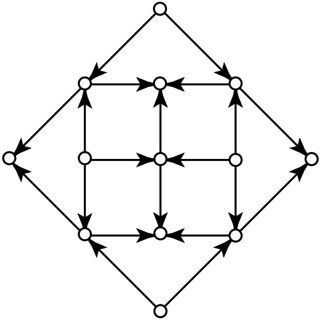

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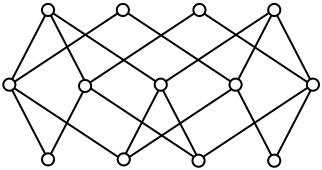

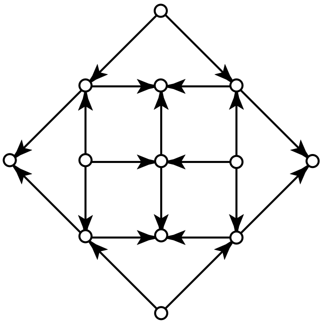

The multifraction, given by the outside boundary, is nontrivial irreducible, nevertheless it represents 1 in the universal group $\Upsilon^{\pm}(P)$.

A sufficient condition for semi-convergence

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The proof (multifractions $\rightsquigarrow$ zigzags) involves the highlighting trick.

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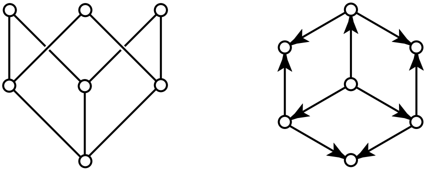

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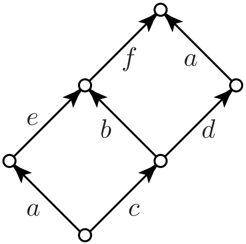

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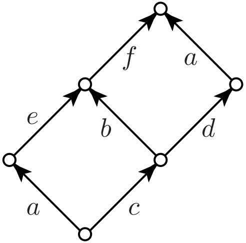

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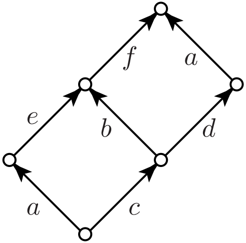

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- Every poset P gives rise to a category $\mathrm{Cat}(P)$, whose elements are the $[x, y]$, where $x \leq y$ in P, and where $[x, y] \cdot [y, z] = [x, z]$. Note $[x, x] = \partial_0[x, y]$ and $[y, y] = \partial_1[x, y]$.

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In particular, for any poset P, $\mathrm{Cat}(P)$ is a gcd-category iff $\mathrm{U_{mon}}(\mathrm{Cat}(P))$ (equal to $\Upsilon(P)$) is a gcd-monoid, iff P is a local lattice.

Categories from extreme spindles

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- Then $\mathrm{Cat}(P, u, v)$ is a category.

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Let $[u,v]$ be an extreme spindle in a local lattice P. Then $\mathrm{Cat}(P,u,v)$ is a gcd-category. Furthermore, $\Upsilon(P,u,v) \underset{\mathrm{def}}{=} \mathrm{U}_{\mathrm{mon}}(\mathrm{Cat}(P,u,v))$ can be defined by the generators $[x,y]$, with $x \leq y$ and $(x,y) \neq (u,v)$, and the relations

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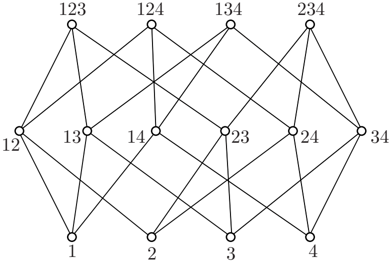
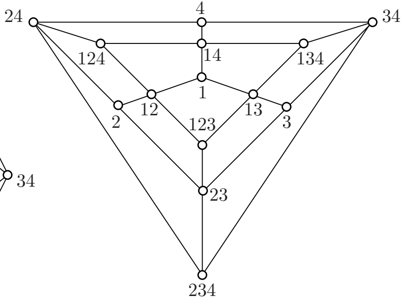
- Setting $u = 1$ and $v = 123$, the interval $[u, v]$ is an extreme spindle of P_B.
- The relation $[1, 12] \cdot [12, 123] = [1, 13] \cdot [13, 123]$ fails in $\Upsilon(P_B, u, v)$, but holds in $\Upsilon^{\pm}(P_B, u, v)$. Hence $\Upsilon(P_B, u, v)$ does not embed into any group.

A picture of P_B





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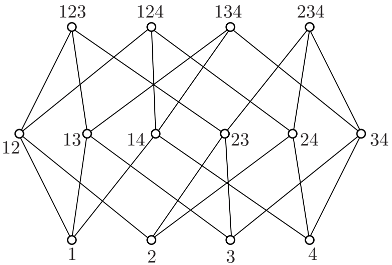

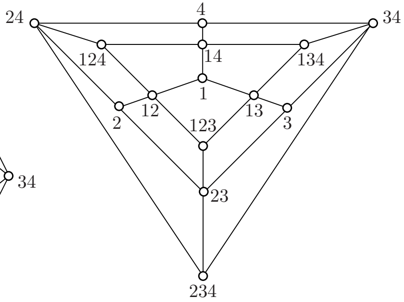

The monoid $\Upsilon(P_B, u, v)$ omits the Mal′cev condition $L_1 R_1 R_2 L_1^* R_2^* L_3 R_1^* R_3 L_3^* L_2^* R_3^*$ (24 variables, $11 + 1$ identities).

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- The result for posets follows trivially (hom-sets are either empty or singletons).

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$$\begin{tikzcd} 0 \arrow[r, bend left, "a"] \arrow[r, "b"] \arrow[r, bend right, "c"'] & 1 \arrow[r, bend left, "a'"] \arrow[r, "b'"] \arrow[r, bend right, "c'"'] & 2 \end{tikzcd}$$

- Let $\psi \colon \mathcal{C}_6 \to \mathbb{Z}^3$ be the unique functor such that $\psi(a) = \psi(a') = (1,0,0)$, $\psi(b) = \psi(b') = (0,1,0)$, $\psi(c) = \psi(c') = (0,0,1)$.
- Then ψ is one-to-one on each hom-set of $\mathcal{C}_6$.
- By the theorem above, C_6 embeds into its group.

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By extending the zigzag machinery to categories, we could prove:

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By extending the zigzag machinery to categories, we could prove:

Proposition (Dehornoy + W)

Reduction is semi-convergent (but not convergent) for C_6.

Thanks for listening!