Categorified operads and cyclic operads

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Symmetric operads

Definition 1 (Partial + non-skeletal)

An *operad* is a functor $\mathcal{O} : \mathbf{Bij}^{op} \to \mathbf{Set}$, together with a distinguished element $id_x \in \mathcal{O}(\{x\})$ and a partial composition operation

$$\circ_x : \mathcal{O}(X) \times \mathcal{O}(Y) \to \mathcal{O}((X \backslash \{x\}) \cup Y).$$

where $x \in X$ and where $X \backslash \{x\}$ and Y *are assumed disjoint*.

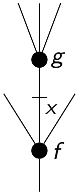
These data are required to satisfy the axioms below.

Parallel associativity.
$(f \circ_x g) \circ_y h = (f \circ_y h) \circ_x g$
Sequential associativity.

$(f \circ_x g) \circ_y h = f \circ_x (g \circ_y h)$
Equivariance.
$f^{\sigma_1} \circ_{\sigma_1^{-1}(x)} g^{\sigma_2} = (f \circ_x g)^{\sigma}$

Left unitality.
$id_y \circ_y f = f$
Right unitality.
$f \circ_x id_x = f.$

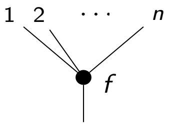
Equivariance (unit).
$id_x{}^{\sigma} = id_u$

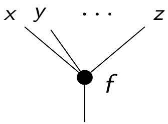


Various styles of definition of (cyclic) operads

	Biased (individual compositions)			**Unbiased**
	Classical	**Partial**		(monad of trees)
Symmetric Operads	*Boardman, Vogt, May*	*Markl*		*Smirnov, May Getzler, Jones*
		Exchangeable output	**Entries only**	
Cyclic Operads	*Getzler, Kapranov*	*Markl*	*Markl*	*Getzler, Kapranov*

Skeletal versus non-skeletal:





More on non-skeletality

We assume given an infinite set V of names (countable is enough). When we stack

- an operation g
- on top of the input labelled x of an operation f whose inputs are labelled (bijectively) by the elements of X
- in view of composing f with g along x,

we make sure that the labels $u, v, \ldots$ of the inputs of g are **fresh** with respect to $X \setminus \{x\}$, i.e. we impose, or "precook" $f \in \mathcal{O}(X)$, $g \in \mathcal{O}(Y)$ so as to have

$$u, v, \ldots \in Y \subseteq V \setminus (X \setminus \{x\})$$

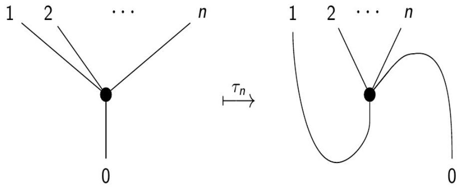
Exchangeable output: from ordinary to cyclic operads

Ordinary operads $\longrightarrow$ **Cyclic operads**

an action of relabeling the leaves of a rooted tree

an action of interchanging the labels of **all** *leaves of a rooted tree, including the label given to the root*

Enriching the operad structure with the action of $\tau_n = (0, 1, \ldots, n)$:



The **distinction between inputs and the output** of an operation is **no longer visible**...

Cyclic operads: entries only

Definition 2 (Partial + non-skeletal)

A *cyclic operad* is a functor $\mathcal{C} : \mathbf{Bij}^{op} \to \mathbf{Set}$,

+ $id_{x,y} \in \mathcal{C}(\{x,y\})$ (for every two elements set)

+ for all X, Y, $x \in X$, $y \in Y$ s.t. $(X \backslash \{x\}) \cap (Y \backslash \{y\})$ are disjoint:

$$ {}_{x}\circ_{y} : \mathcal{C}(X) \times \mathcal{C}(Y) \to \mathcal{C}((X \backslash \{x\}) \cup (Y \backslash \{y\})). $$

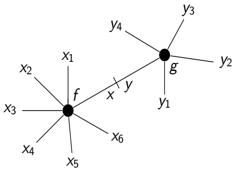
These data are required to satisfy the axioms below.

Parallel associativity.
$(f \,{}_{x}\circ_{y}\, g) \,{}_{u}\circ_{z}\, h = (f \,{}_{u}\circ_{z}\, h) \,{}_{x}\circ_{y}\, g$

Equivariance *(composition).*
$f^{\sigma_1} \,{}_{x}\circ_{y}\, g^{\sigma_2} = (f \,{}_{\sigma_1(x)}\circ_{\sigma_2(y)}\, g)^{\sigma}$

Left unitality.
$id_{x,y} \,{}_{y}\circ_{x}\, f = f$

Equivariance (unit).
$id^{\sigma}_{x,y} = id_{u,v}.$



Properties:

Commutativity.
$f \,{}_{x}\circ_{y}\, g = g \,{}_{y}\circ_{x}\, f$

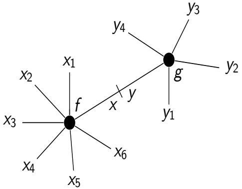
Sequential associativity.
$(f \,{}_{x}\circ_{y}\, g) \,{}_{u}\circ_{z}\, h = f \,{}_{x}\circ_{y}\, (g \,{}_{u}\circ_{z}\, h)$

Right unitality.
$f \,{}_{x}\circ_{y}\, id_{x,y} = f$

Cyclic operads: unbiased definition

The *entries-only* characterization of cyclic operads reflects the ability to carry out the (partial) composition of two operations along *any* edge.

> The pasting shemes for cyclic operads are unrooted, decorated, labeled trees.



Given $\mathcal{P} : \mathbf{Bij}^{op} \to \mathbf{C}$, we build the *free cyclic operad* $F(\mathcal{P})$ by grafting of such trees. The free operad functor F and the forgetful functor U constitute a monad $\Gamma = UF$ in $\mathbf{C}^{\mathbf{Bij}^{op}}$, called the *monad of unrooted trees*.

Definition 3

A cyclic operad is an algebra over this monad.

Coherence theorems in category theory

In order to ensure that all diagrams made of canonical arrows commute, it suffices to check a small number of commutations.

Mac Lane: MONOIDAL CATEGORY

$$\mathbf{C},\ \otimes : \mathbf{C} \times \mathbf{C} \to \mathbf{C},\ \beta : (f \otimes g) \otimes h \to f \otimes (g \otimes h)$$

Coherence of monoidal categories: If the PENTAGON commutes, then all diagrams made of β-arrows commute.

$$\begin{tikzcd}
 & ((fg)h)k \arrow[dl, "\beta \cdot 1"'] \arrow[dr, "\beta"] & \\
(f(gh))k \arrow[d, "\beta"'] & & (fg)(hk) \arrow[d, "\beta"] \\
f((gh)k) \arrow[rr, "1 \cdot \beta"] & & f(g(hk))
\end{tikzcd}$$

Coherence theorems in category theory

In order to ensure that all diagrams made of canonical arrows commute, it suffices to check a small number of commutations.

Mac Lane: SYMMETRIC MONOIDAL CATEGORY

$$\mathbf{C},\ \otimes : \mathbf{C}\times\mathbf{C}\to\mathbf{C},\ \beta : (f\otimes g)\otimes h \to f\otimes(g\otimes h),\quad c : f\otimes g\to g\otimes f$$

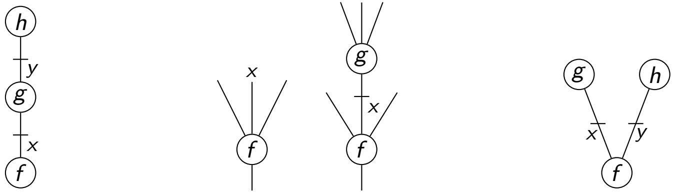
Coherence of symmetric monoidal categories: If the PENTAGON and the HEXAGON commute, then all *linear* diagrams made of β- and c-arrows commute.

$$\begin{tikzcd}
 & ((fg)h)k \arrow[dl, "\beta\cdot 1"'] \arrow[dr, "\beta"] & \\
(f(gh))k \arrow[d, "\beta"'] & & (fg)(hk) \arrow[d, "\beta"] \\
f((gh)k) \arrow[r, "1\cdot\beta"] & f(g(hk)) &
\end{tikzcd}
\qquad
\begin{tikzcd}
(fg)h \arrow[r, "\beta"] \arrow[d, "c\cdot 1"'] & f(gh) \arrow[r, "c"] & (gh)f \arrow[d, "\beta"] \\
(gf)h \arrow[r, "\beta"] & g(fh) \arrow[r, "1\cdot c"] & g(hf)
\end{tikzcd}$$

Coherence theorems in operad theory: weakening the associativity of operadic composition

Operad (non-unital): a functor $\mathcal{O} : \mathbf{Bij}^{op} \to \mathbf{Set}$, together with *insertions* $\circ_x : \mathcal{O}(X) \times \mathcal{O}(Y) \to \mathcal{O}(X \backslash \{x\} \cup Y)$, such that

$$(f \circ_x g) \circ_y h = f \circ_x (g \circ_y h) \qquad \text{and} \qquad (f \circ_x g) \circ_y h = (f \circ_y h) \circ_x g.$$



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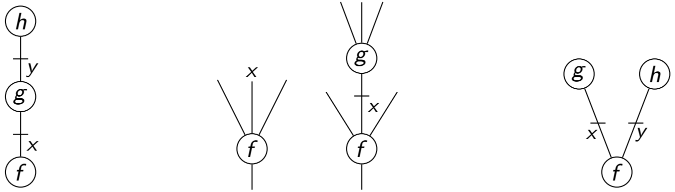

Cat-operad: **Set** replaced by **Cat**

Coherence theorems in operad theory: weakening the associativity of operadic composition

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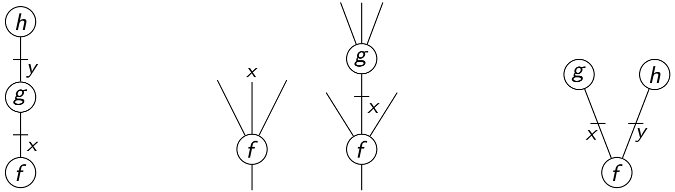

Cat-operad: **Set** replaced by **Cat**

Došen and Petrić (2015):

Weak Cat-operad: associativity equations replaced by isomorphisms $\beta : (f \circ_x g) \circ_y h \to f \circ_x (g \circ_y h)$ and $\theta : (f \circ_x g) \circ_y h \to (f \circ_y h) \circ_x g$

Coherence conditions of Weak Cat-operads

(β-pentagon)

$$\begin{tikzcd}
& ((fg)h)k \arrow[dl, "\beta\cdot 1"'] \arrow[dr, "\beta"] & \\
(f(gh))k \arrow[d, "\beta"'] & & (fg)(hk) \arrow[d, "\beta"] \\
f((gh)k) \arrow[rr, "1\cdot\beta"] & & f(g(hk))
\end{tikzcd}$$

($\beta\theta$-hexagon)

$$\begin{tikzcd}
((fg)h)k \arrow[r, "\theta"] \arrow[d, "\beta\cdot 1"'] & ((fg)k)h \arrow[d, "\beta\cdot 1"] \\
(f(gh))k \arrow[d, "\beta"'] & (f(gk))h \arrow[d, "\beta"] \\
f((gh)k) \arrow[r, "1\cdot\theta"] & f((gk)h)
\end{tikzcd}$$

(θ-hexagon)

$$\begin{tikzcd}
((fg)h)k \arrow[r, "\theta\cdot 1"] \arrow[d, "\theta"'] & ((fh)g)k \arrow[d, "\theta"] \\
((fg)k)h \arrow[d, "\theta\cdot 1"'] & ((fh)k)g \arrow[d, "\theta\cdot 1"] \\
((fk)g)h \arrow[r, "\theta"] & ((fk)h)g
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((fh)g)k \arrow[d, "\beta"'] & & ((fg)k)h \arrow[d, "\beta\cdot 1"] \\
(fh)(gk) \arrow[rr, "\theta"] & & (f(gk))h
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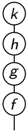

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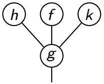

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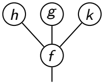

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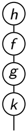

Weak cyclic Cat-operad: the definition

A **weak cyclic Cat-operad** (non-unital): a functor $\mathcal{C} : \mathbf{Bij}^{op} \to \mathbf{Cat}$, +

- insertions ${}_x\circ_y : \mathcal{C}(X) \times \mathcal{C}(Y) \to \mathcal{C}(X\backslash\{x\} \cup Y\backslash\{y\})$, and
- and a familly of natural isomorphisms

$$\beta : (f \;{}_x\circ_y\; g) \;{}_u\circ_z\; h \to f \;{}_x\circ_y\; (g \;{}_u\circ_z\; h) \quad \text{and} \quad c : f \;{}_x\circ_y\; g \to g \;{}_x\circ_y\; f$$

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$$\begin{tikzcd}
 & (h(fg))k \arrow[r, "\beta"] & h((fg)k) \arrow[dr, "c"] & \\
((fg)h)k \arrow[ur, "c\cdot 1"] \arrow[rrr, "\theta"] \arrow[d, "\beta\cdot 1"'] & & & ((fg)k)h \arrow[d, "\beta\cdot 1"] \\
(f(gh))k \arrow[d, "\beta"'] & & & (f(gk))h \arrow[d, "\beta"] \\
f((gh)k) \arrow[dr, "1\cdot c\cdot 1"] & & & f((gk)h) \\
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\end{tikzcd}$$

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(fg)h \arrow[r, "\beta"] \arrow[d, "c\cdot 1"'] & f(gh) \arrow[r, "c"] & (gh)f \arrow[d, "c\cdot 1"] \\
(gf)h \arrow[r, "c"] & h(gf) & (hg)f \arrow[l, "\beta"']
\end{tikzcd}
\qquad
\begin{tikzcd}
fg \arrow[d, "c"'] \arrow[dr, "1"] & \\
gf \arrow[r, "c"] & fg
\end{tikzcd}$$

$$\beta^{\sigma}_{f,g,h} = \beta_{f^{\sigma_1}, g^{\sigma_2}, h^{\sigma_3}}$$
$$c^{\sigma}_{f,g} = c_{f^{\sigma_1}, g^{\sigma_2}}$$
$$(\varphi \cdot \psi)^{\sigma} = \varphi^{\sigma_1} \cdot \psi^{\sigma_2}$$

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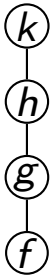

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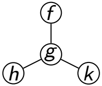

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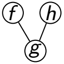

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Coherence: a formal language

- Object terms:

$$\mathcal{W} ::= \underline{a} \mid (\mathcal{W}\,{}_x\square_y\,\mathcal{W}) \mid \mathcal{W}^\sigma$$

with (in the second rule, $x \in X$, $y \in Y$, and $(X\backslash\{x\}) \cap (Y\backslash\{y\}) = \emptyset$):

$$\frac{a \in \mathcal{C}(X)}{\underline{a} : X} \qquad \frac{\mathcal{W}_1 : X \quad \mathcal{W}_2 : Y}{\mathcal{W}_1\,{}_x\square_y\,\mathcal{W}_2 : (X\backslash\{x\}) \cup (Y\backslash\{y\})} \qquad \frac{\mathcal{W} : X \quad \sigma : Y \to X}{\mathcal{W}^\sigma : Y}$$

- Arrow terms: $\Phi ::=$

$$1_{\mathcal{W}} \mid \beta^{x,\underline{x};y,\underline{y}}_{\mathcal{W}_1,\mathcal{W}_2,\mathcal{W}_3} \mid c^{x,y}_{\mathcal{W}_1,\mathcal{W}_2} \mid \varepsilon 1^{\sigma}_{\underline{a}} \mid \varepsilon 2_{\mathcal{W}} \mid \varepsilon 3^{\sigma,\tau}_{\mathcal{W}} \mid \varepsilon 4^{x,y;x',y'}_{\mathcal{W}_1,\mathcal{W}_2;\sigma} \mid \Phi \circ \Phi \mid \Phi\,{}_x\square_y\,\Phi \mid \Phi^\sigma$$

(plus inverses of β and the ϵ_i's) with, say:

$$\frac{}{\varepsilon 1^{\sigma}_{\underline{a}} : \underline{a}^\sigma \to \underline{a^\sigma}} \qquad \frac{}{\varepsilon 2_{\mathcal{W}} : \mathcal{W}^{id_X} \to \mathcal{W}} \qquad \frac{}{\varepsilon 3^{\sigma,\tau}_{\mathcal{W}} : (\mathcal{W}^\sigma)^\tau \to \mathcal{W}^{\sigma\circ\tau}}$$

$$\frac{}{\varepsilon 4^{x,y;x',y'}_{\mathcal{W}_1,\mathcal{W}_2;\sigma} : (\mathcal{W}_1\,{}_x\square_y\,\mathcal{W}_2)^\sigma \to \mathcal{W}_1^{\sigma_1}\,{}_{x'}\square_{y'}\,\mathcal{W}_2^{\sigma_2}} \qquad \frac{\varphi_1 : W_1 \to W_2 \quad \varphi_2 : W_2 \to W_3}{\varphi_2 \circ \varphi_1 : W_1 \to W_3}$$

all interpreted obviously in $\mathcal{C}$, setting $[[\varepsilon_1]], [[\varepsilon_2]], [[\varepsilon_3]], [[\varepsilon_4]]$ to be the identity (in our setting, *equivariance is NOT weakened*).

Coherence:the statement

We note that if $\Phi : \mathcal{W}_1 \rightarrow \mathcal{W}_2$ then $\mathcal{W}_1 : X$ and $\mathcal{W}_2 : X$ for some X.

Coherence theorem

For any pair of parallel arrow terms $\Phi, \Psi : \mathcal{W}_1 \rightarrow \mathcal{W}_2$, we have

$$[[\Phi]] = [[\Psi]]$$

in $\mathcal{C}(X)$.

Plan of the proof

Reduce the proof to the coherence for (non-symmetric, skeletal) operads (Došen and Petrić):

- First reduction: getting rid of the actions σ.
- Second reduction: operadic "make-up" ($=$ reduction from "cyclic" to "just operadic")
- Third reduction: skeletisation (assigning a total order on the inputs of each involved operation)

First reduction (on object terms)

We (weakly) normalise every object term to one in the sub-syntax

$$W ::= \underline{a} \mid (W\,{}_x\square_y\,W)$$

by the following inductive definition:

$$\frac{}{\underline{a} \rightsquigarrow \underline{a}} \qquad \frac{\mathcal{W}_1 \rightsquigarrow W_1 \qquad \mathcal{W}_2 \rightsquigarrow W_2}{\mathcal{W}_1{}_x\square_y\mathcal{W}_2 \rightsquigarrow W_1{}_x\square_y W_2}$$

$$\frac{}{\underline{a}^\sigma \rightsquigarrow \underline{a}^\sigma} \qquad \frac{\mathcal{W} \rightsquigarrow W}{\mathcal{W}^{id_X} \rightsquigarrow W} \qquad \frac{\mathcal{W}^{\sigma\circ\tau} \rightsquigarrow W}{(\mathcal{W}^\sigma)^\tau \rightsquigarrow W}$$

$$\frac{\mathcal{W}_1^{\sigma_1} \rightsquigarrow W_1 \quad \mathcal{W}_2^{\sigma_2} \rightsquigarrow W_2}{(\mathcal{W}_1\,{}_x\square_y\,\mathcal{W}_2)^\sigma \rightsquigarrow W_1\,{}_{x'}\square_{y'}\,W_2}$$

This is non-deterministic (choice of fresh x', y' in the last rule), but:

$$\mathcal{W} \rightsquigarrow W_1\ ,\ \mathcal{W} \rightsquigarrow W_2 \quad \Rightarrow \quad W_1 \equiv W_2$$

where $\equiv$ ("α-conversion") is defined by the axiom

$$\frac{}{W_1\,{}_x\square_y\,W_2 \equiv W_1[\underline{a^{\tau_1}}/\underline{a}]\,{}_{x'}\square_{y'}\,W_2[\underline{b^{\tau_2}}/\underline{b}]}$$

First reduction (on arrow terms)

Similarly, one defines a normalisation function on arrow terms. Here are a few cases:

$$\frac{\mathcal{W}_i \rightsquigarrow W_i \quad i \in \{1,2,3\}}{\beta^{x,\underline{x};y,\underline{y}}_{\mathcal{W}_1,\mathcal{W}_2,\mathcal{W}_3} \rightsquigarrow \beta^{x,\underline{x};y,\underline{y}}_{W_1,W_2,W_3}}$$

$$\frac{}{\varepsilon 1^{\sigma}_{\underline{a}} \rightsquigarrow 1_{\underline{a^{\sigma}}}} \qquad \frac{\mathcal{W} \rightsquigarrow W}{\varepsilon 2_{\mathcal{W}} \rightsquigarrow 1_W} \qquad \text{and similarly for } \epsilon_3 \text{ and } \epsilon_4$$

$$\frac{\Phi_1^{\sigma_1} \rightsquigarrow \varphi_1 \quad \Phi_2^{\sigma_2} \rightsquigarrow \varphi_2}{(\Phi_1 \,{}_x\square_y\, \Phi_2)^{\sigma} \rightsquigarrow \varphi_1 \,{}_{x'}\square_{y'}\, \varphi_2} \qquad \frac{\Phi_1^{\sigma} \rightsquigarrow \varphi_1 \quad \Phi_2^{\sigma} \rightsquigarrow \varphi_2}{(\Phi_2 \circ \Phi_1)^{\sigma} \rightsquigarrow \varphi_2 \circ \varphi_1}$$

and (leaving some indices out):

$$\frac{\mathcal{W}_1^{\sigma_1} \rightsquigarrow W_1 \quad \mathcal{W}_1^{\sigma_2} \rightsquigarrow W_2 \quad \mathcal{W}_3^{\sigma_1} \rightsquigarrow W_3}{\beta^{\sigma}_{\mathcal{W}_1,\mathcal{W}_2,\mathcal{W}_3} \rightsquigarrow \beta_{W_1,W_2,W_3}}$$

But there are two subtleties.

Subtlety one: a typing issue

In the rule defining $(\Phi_2 \circ \Phi_1)^\sigma \rightsquigarrow \varphi_2 \circ \varphi_1$, the term $\varphi_2 \circ \varphi_1$ *might not type-check*. It does only with the following *more liberal* typing rule for arrow term composition:

$$\frac{\vdash \varphi_1 : W_1 \to W_2 \quad \vdash \varphi_2 : W_2' \to W_3 \quad W_2 \equiv W_2'}{\vdash \varphi_2 \circ \varphi_1 : W_1 \to W_3}$$

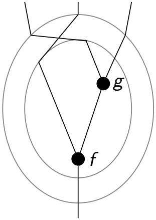
But this is OK, because this rule is "admissible":

• If $\vdash \varphi : U \to V$ in the more liberal system and if $U \equiv U'$,
then there exists V' and φ' (uniquely determined by φ and U') such that
• $V' \equiv V$ and $\varphi' : U' \to V'$ in the original system.

The reduction $\mathtt{red}_1$ is thus in fact the composition of two reductions:

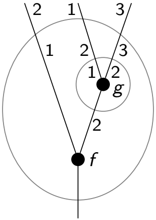
1) the reduction of the previous slide (tenamed as $\mathtt{red}_{11}$),
2) the reduction $\mathtt{red}_{12}$ from φ to φ'.

Subtlety two: the first reduction needs the skeletal setting

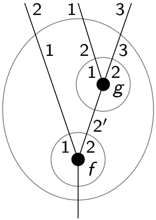


$\xrightarrow{\text{skeletal}}$?

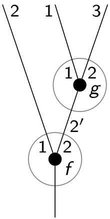
$(f \circ_2 g)^{\binom{2\,1\,3}{1\,2\,3}}$



$=$



$\xrightarrow{\text{non-skeletal}}$



$(f \circ_2 g^{\binom{2\,3}{1\,2}})^{\binom{2\,1\,3}{1\,2\,3}} =_\alpha (f^{\binom{1\,2'}{1\,2}} \circ_{2'} g^{\binom{2\,3}{1\,2}})^{\binom{2\,1\,3}{1\,2\,3}} =_{\mathtt{EQ}} (f^{\binom{1\,2'}{1\,2}})^{\binom{2\,2'}{1\,2'}} \circ_{2'} (g^{\binom{2\,3}{1\,2}})^{\binom{1\,3}{2\,3}} = f^{\binom{2\,2'}{1\,2}} \circ_{2'} g^{\binom{1\,3}{1\,2}}$

Second reduction

If $\varphi : W_1 \to W_2$ (for $W_1, W_2 : X$), then W_1 and W_2 share not only their set of free names, but also their *underlying unrooted tree* (cf. unbiased definition of cyclic operad).
By picking a free (half-)edge $x \in X$ of the tree, one gets a *rooted tree*, which dictates the definition of an "operadic make-up" and one gets

$$\kappa(X,x)(W) : W \to \mathtt{red}_2(X,x)(W)$$
$$\mathtt{red}_2(X,x)(\varphi) : \mathtt{red}_2(X,x)(W_1) \to \mathtt{red}_2(X,x)(W_2)$$

$$[[\mathtt{red}_2(X,x)(\varphi)]] \circ [[\kappa(X,x)(W_1)]] = [[\kappa(X,x)(W_2)]] \circ [[\varphi]]$$

where $\mathtt{red}_2(X,x)(\varphi)$ is an arrow term of the formal language

$$\alpha ::= 1 \mid \beta^{x,\underline{x};y,\underline{y}}_{W_1,W_2,W_3} \mid {\beta^{x,\underline{x};y,\underline{y}}_{W_1,W_2,W_3}}^{-1} \mid \theta^{x,\underline{x};y,\underline{y}}_{W_1,W_2,W_3} \mid \alpha \circ \alpha \mid \alpha\, {}_x\square_y\, \alpha$$

which is now (weak) *operadic* (as opposed to cyclic operadic).
($[[\theta]]$ is defined as $c \circ \beta \circ (c \square 1)$ in $\mathcal{C}$)

Third reduction

We associate with $\mathcal{C}$ a *skeletal, non-symmetric weak Cat-operad* $\mathcal{O}_{\mathcal{C}}$:

- Objects of $\mathcal{O}_{\mathcal{C}}(n)$: quadruples (X, x, σ, f), where $|X| = n+1$, $x \in X$, $f \in \mathcal{C}(X)$ and $\sigma : [n] \to X \backslash \{x\}$ is a bijection (inducing a total order on $X \backslash \{x\}$).
- $\mathcal{O}_{\mathcal{C}}(n)[(X, x, \sigma, f), (X, x, \sigma, g)] = \{(X, x, \sigma, \alpha) \in \mathcal{C}(X)[f, g] \mid \}$.

Now, given $\alpha : W_1 \to W_2$ as produced by the second reduction, we can further *planarise* the underlying rooted tree $\mathcal{T}$ of W_1, W_2:

- Assign a total order $\sigma_a : [n] \to X$ to every node $a \in \mathcal{C}(X)$ in $\mathcal{T}$.

Calling $\vec{\sigma}$ this collection of additional data, we finally define $\mathtt{red}_3(X, x, \mathcal{T}, \vec{\sigma})(\alpha)$ such that:

$$[[\mathtt{red}_3(X, x, \mathcal{T}, \vec{\sigma})(\alpha)]]_{\mathcal{O}_{\mathcal{C}}} \;=\; (X, x, \tau, [[\alpha]]_{\mathcal{C}})$$

where $\tau : [m] \to X \setminus \{x\}$ is the global total ordering induced by the σ_a's.

Assembling the puzzle

The first reduction is designed in such a way that we have:

- $[[\mathtt{red}_1(\Phi)]] = [[\Phi]]$

Moreover, from the previous slides we have:

- $[[\mathtt{red}_2(X,x)(\varphi_1)]] = [\mathtt{red}_2(X,x)(\varphi_2)]] \;\; \Rightarrow \;\; [[\varphi_1]] = [[\varphi_2]]$
- $[[\mathtt{red}_3(X,x,\mathcal{T},\vec{\sigma})(\alpha)]] = [[\mathtt{red}_3(X,x,\mathcal{T},\vec{\sigma})(\beta)]] \;\; \Rightarrow \;\; [[\alpha]] = [[\beta]]$

If Φ, Ψ are parallel, then their reductions are parallel:

$$\mathtt{red}_3(X,x,\mathcal{T},\vec{\sigma})(\mathtt{red}_2(X,x)(\mathtt{red}_1(\Phi))) \text{ and}$$
$$\mathtt{red}_3(X,x,\mathcal{T},\vec{\sigma})(\mathtt{red}_2(X,x)(\mathtt{red}_1(\Psi)))$$

Then we conclude

by the coherence result of Došen and Petrić

$+$ the three items above.

An example: generalised profunctors

- Given a category $\mathbf{D}$ with enough colimits
- equipped with a duality $(-)^* : \mathbf{D}^{op} \to \mathbf{D}$,

a categorified cyclic operad $\mathcal{C} : \Sigma^{op} \to \mathbf{Cat}$ can be obtained:

- $\mathcal{C}(X) = \int^n [\mathbf{D}^n, \mathbf{Set}] \times \mathbf{Bij}[n, X]$ (a generalised form of profunctor) (thus an operation is an equivalence class $[(F, \phi)]$
- with operadic composition given by

$$[(F,\phi)]\,{}_x\circ_y[(G,\psi)] \;=\; [(F\,{}_{\phi^{-1}(x)}\circ_{\psi^{-1}(y)}G, \chi)]$$

where, say (for $F : \mathbf{D}^3 \to \mathbf{Set}$ and $G : \mathbf{D}^2 \to \mathbf{Set}$):

$$(F\,{}_2\circ_1\,G)(a,c,e) = \int^{b,d} F(a,b,c) \times G(d,e) \times [b,d],$$

where $[-,-] : D^{op} \times D^{op} \to \mathbf{Set}$ is defined by $[x,y] = D[x,y^*]$, and where χ is defined by

$$\begin{array}{ll} \chi(i- = \phi(i)\ (i < \phi^{-1}(x)) & \chi(i) = \phi(i+1)\ (\phi^{-1}(x) \leq i \leq m-1) \\ \chi(m-1+j) = \psi(j)\ (j < \psi^{-1}(y)) & \chi(m-1+j) = \psi(j+1)\ (\psi^{-1}(y) \leq j \leq n-1) \end{array}$$

Skeletal coherence

We can also formulate coherence conditions for categorified cyclic operads

- in the exchancheable-output, non-skeletal setting,
- in the skeletal (and then necessarily exhangeable-output) setting.

The proof of coherence is obtained by adding the following further reductions:

- define a non-skeletal exchangeable-output categorified cyclic operad from the skeletal one and reduce coherence to coherence in this new structure;
- define a non-skeletal entries-only categorified cyclic operad from the non-skeletal exchangeable-output one, and reduce coherence to coherence in this new structure.

The good side of non-skeletality

- It allows to display the entries-only presentation of cyclic operads (no such thing as commutativity with numbered wires!)
- Non-skeletality turns out to be crucial for the rewriting involved in our proof of coherence in the presence of symmetries. (In particular, skeletal coherence in the presence of symmetries has to be first reduced to non-skeletal one!)

Thank you!